

Mario Pieri's View of the Symbiotic Relationship between the Foundations and the Teaching of Elementary Geometry in the Context of the Early Twentieth Century Proposals for Pedagogical Reform

Elena Anne Corie Marchisotto, Ana Millán Gasca

IN **PHILOSOPHIA SCIENTIÆ** 2021/1 25-1 , PAGES 157 TO 183

PUBLISHER **UNIVERSITÉ DE LORRAINE**

ISSN 1281-2463

Uploaded: 05/28/2021

Article available online at

<https://shs.cairn.info/journal-philosophia-scientiae-2021-1-page-157?lang=en>



Discover the contents of this issue, follow the journal by email, subscribe...
Scan this QR code to access the page for this issue on Cairn.info.



Electronic distribution Cairn.info for Université de Lorraine.

You are authorized to reproduce this article within the limits of the terms of use of Cairn.info or, where applicable, the terms and conditions of the license subscribed to by your institution. Details and conditions can be found at cairn.info/copyright.

Unless otherwise provided by law, the digital use of these resources for educational purposes is subject to authorization by the Publisher or, where applicable, by the collective management organization authorized for this purpose. This is particularly the case in France with the CFC, which is the approved organization in this area.

Mario Pieri's View of the Symbiotic Relationship between the Foundations and the Teaching of Elementary Geometry in the Context of the Early Twentieth Century Proposals for Pedagogical Reform

Elena Anne Corie Marchisotto

Department of Mathematics,
California State University, Northridge (USA)

Ana Millán Gasca

Department of Education, Università Roma Tre (Italy)

Résumé : Dans cet article nous discutons les enjeux des essais de réforme de l'enseignement de la géométrie élémentaire au tournant du XIX^e siècle en examinant la contribution de Mario Pieri, membre distingué des entourages de Peano et Segre à Turin. Le rapport symbiotique entre axiomatique et pédagogie et le rôle de l'intuition dans l'apprentissage sont deux aspects majeurs à cet égard. La notion d'intuition a été l'objet d'un grand nombre d'analyses qui ont une longue histoire, de la part de philosophes, mathématiciens, didacticiens des mathématiques, psychologues, historiens. Pour établir le contexte de la réflexion de Pieri, on cherche tout simplement à offrir un bref aperçu de quelques idées au sujet de son rôle pédagogique, dans l'instruction élémentaire et jusqu'à celle universitaire, qui peuvent avoir façonné les efforts pur l'amélioration de l'enseignement de la géométrie au secondaire au début du XX^e siècle. Pieri prend en compte la question de l'intuition dans plusieurs parmi ses travaux d'axiomatisation, et notamment dans ceux qui sont consacrés à la géométrie projective qui a été son intérêt principal du point de vue des fondements des mathématiques. On considère ici cependant surtout ses systèmes axiomatiques pour la géométrie élémentaire, où il adopte l'approche basée sur les transformations géométriques de Felix Klein. Notre objectif est de représenter la pensée de Pieri sur la façon d'intégrer deux types d'intuition,

dites sensible et rationnelle, dans les démarches pour l'amélioration de l'enseignement de la géométrie d'Euclide. Nous montrons comment les vues de Pieri sur l'intuition géométrique et la réforme pédagogique ont été soit ignorées, soit déformées dans plusieurs publications notables au tournant du XX^e siècle. En particulier, nous donnons à Pieri une voix en réponse aux commentaires spécifiques formulés au début des années 1900 par Federigo Enriques, Ugo Amaldi et Florian Cajori dans des publications largement diffusées inspirées de Klein.

Abstract: In this paper, we discuss a proposal for reform in the teaching of Euclidean geometry that reveals the symbiotic relationship between axiomatics and pedagogy. We examine the role of intuition in this kind of reform, as expressed by Mario Pieri, a prominent member of the Schools of Peano and Segre at the University of Turin. We are well aware of the centuries of attention paid to the notion of intuition by mathematicians, mathematics educators, philosophers, psychologists, historians, and others. To set a context for Pieri's proposal, we only seek to open a small window on views of the pedagogical role of intuition, from primary education to university study that may have informed early 20th century efforts to improve the teaching of geometry at the secondary school level. Pieri addressed the topic of intuition in many of his axiomatizations, including those in projective geometry which was his main area of concentration in foundations. We focus here primarily on his axiom systems for elementary geometry, which embraced the transformational approach of Felix Klein's vision for the subject. Our goal is to convey Pieri's thoughts on how to integrate two types of intuition, denoted as *sensible* and *rational*, in his endeavors to improve the teaching of the geometry of Euclid. We show how Pieri's views on geometric intuition and pedagogical reform were either ignored or misrepresented in several notable publications at the turn of the 20th century. In particular, we give Pieri a voice in response to specific comments made in the early 1900s by Federigo Enriques, Ugo Amaldi, and Florian Cajori in widely circulated publications inspired by Klein.

1 Introduction

In this paper, we examine perceptions of the role of intuition in certain early 20th century proposals for reform of Euclidean geometry. We begin with a discussion of *sensible* intuition and studies in early education that promoted the teaching of intuitive geometry. Our examination of pedagogical reforms emanating from considerations of *rational* intuition forms the basis for our discussion of the relationship between the foundations of geometry and its teaching, as expressed by Mario Pieri in the context of his membership of the Schools of Giuseppe Peano and Corrado Segre at the University of

Turin. We show how Pieri's ideas were ignored or misrepresented by certain notable scholars who recognized the close connection between mathematical formulation and its educational outcomes, and by others who harbored deep rooted prejudices against that connection. In this short paper, we limit our focus to comments about Pieri in three publications inspired by Felix Klein that were widely circulated in the early decades of the 1900s.¹

2 Sensible intuition as an impetus for pedagogical reform: Intuitive geometry

Before the 19th century, “geometry” was essentially a synonym for Euclidean geometry. The *Elements* served as a prism through which to view the subject. Synthetic geometry, in the style of Euclid, dominated the mathematical curriculum, both as a foundation for and educational pathway to mathematics [Giusti 1993, 2], [Rowe 2018, 370].

The teaching of synthetic geometry was considered gymnastics for the mind that could only be fostered by a system of propositions as those set by Euclid. [Barbin & Menghini 2014, 482]

The use of the *Elements* in the classroom throughout history, for its content and methodology, is complicated to describe, in part due to variations in when and how it was adopted in the educational spectrum.² There is no question however, that well before the turn to the 20th century, serious concerns had emerged about its efficacy as a basis for geometry textbooks. There were various reasons for this. Among them was the exclusion of algebraic methods in Euclid's closed system, criticized by those who advocated a more analytic approach to the subject. Another was the belief that the emphasis on a formal exposition of geometry neglects the psychological and emotional needs of students, neither capturing their interest nor encouraging them to think mathematically.

The quest for pedagogical reform drew from a wide spectrum of both mathematics, which included research on its foundations, and mathematics education, in studies that ran the gamut from primary school to university. Scores of textbooks written with the intention of improving the exposition

1. For broader discussions of Pieri's pedagogy in the context of his tenure at the University of Turin, see *Quaderni di storia dell'Università di Torino*, 10 (2009-2011), <https://www.omeka.unito.it/omeka/items/show/396>; also [Marchisotto 2010], [Luciano 2012a,b, 2017], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 10].

2. For the most part, academic geometry textbooks had begun to find a place in European secondary schools in the 18th century. For comprehensive discussions see [Karp & Schubring 2014, Section IV].

of Euclid would appear. Among them, editions of [Clairaut 1741-1852] and [Legendre 1752-1833] found their place in 19th century classrooms. That their authors were noted research mathematicians, Alexis Clairaut and Adrien-Marie Legendre, is indicative of a history of concern about pedagogical issues within the mathematical community. Applying strategies of practical geometry, Clairaut replaced Euclid's deductive proofs with geometric constructions and reasoning based on them.³ Legendre instead embraced Euclid's deductive methodology, but infused it with arithmetic notation and appeals to intuition. His approach was metrical as he envisioned geometry as a "science that has as its object the measure of extent"⁴ [Legendre 1752-1833, citation from 1823, 1, Definition VII]. Legendre's text, which was widely used in Italy until 1867, inspired a great number of others, many less rigorous than his, and replete with mistakes. Largely through advocacy of Luigi Cremona, the noted mathematician who devoted himself to the study of geometry and its teaching, an Italian translation of Euclid's *Elements* [Betti & Brioschi 1867] was adopted for the schools. In response to those who believed that the return to a more rigorous text would be too challenging for "less gifted" students, Cremona reminded the Italian public that in Germany there was an increasing number of geometry books designed to be "more accessible to even mediocre intellects" [Cremona 1873, vii], see [Giacardi 2012, Giacardi & Scoth 2014], [Millán Gasca 2011], [Israel 2017].

Concerted national and international efforts had emerged by the turn to the 20th century to address the reform of secondary school mathematics teaching. Notable among them was the *Commission internationale de l'enseignement mathématique* whose founding president was Klein. Intuition in geometry became a focus for discussion.⁵ Certain proposals called for the teaching of "intuitive geometry", which promoted strategies that emphasize "informal reasoning", understood as argumentation based on observation for which justifications are not explicitly provided. At the heart of this method of teaching is the idea of *sensible* intuition, as a form of immediate knowing, possibly linked to data provided by the senses from the very presence of the object of knowledge, see [Betz 1933], [Hendrix 1936].

There were many who advocated the teaching of intuitive geometry well before secondary school.⁶ The Swiss education reformer, Johann Pestalozzi, a teacher of the geometer Jacob Steiner, believed that the education of the *mind* (the *head*, in his terminology) included three integrated components, *word*, *number*, and *form* [Pestalozzi, Cook *et al.* 1894, 118]. The teaching

3. Along with geometric constructions, practical geometry incorporated mensuration techniques and other problem-solving strategies rooted in arts and crafts. It was adopted in middle school education, see [Menghini 2015].

4. This translation is ours, as are all others unless otherwise indicated.

5. For example [Poincaré 1899]. By this time, various interpretations of "intuition" had emerged from philosophical investigations, see [Osbeck & Held 2014].

6. For a discussion of developments in the teaching of geometry to children, see [Millán Gasca 2015].

of geometry (*form*) based on drawing is propaedeutic to graphical aspects of writing and reading, and is therefore central to early education.

The use of diagrams was intimately linked to intuitive reasoning, see [Lorenat 2020]. Students were taught linear drawing to initiate them into the study of geometry, see [D'Enfert 2003]. The German educator, Friedrich Fröbel, who was another of Pestalozzi's students, created a collection of geometrical exercises of decomposition and ratio with 3-dimensional regular forms for his kindergarten students [Fröbel 1826]. William George Spencer⁷ wrote *Inventional Geometry* [Spencer 1860], which consisted of Pestalozzian sequences of questions appealing to intuition, designed to familiarize the pupil with geometrical concepts.

Insights about cognition obtained by studying early childhood education suggested other pathways for learning. Texts such as *Geometry for Beginners* [Minchin 1898] by the Irish mathematician and physicist George M. Minchin addressed the roles of instinct and observation in learning. Writers like Mary Everest Boole, wife of logician George Boole, stressed the need to prepare the unconscious mind for the development of a scientific attitude in children by “restoring the vitality of geometric instinct” [Boole 1904, 68]. Édouard Séguin, who pioneered modern educational methods for teaching cognitively-impaired children, considered the motion of the finger or the pencil from one point to another point “avec rectitude et précision” [Séguin 1843, citation from 1897, 123] as the first law of understanding. His ideas to promote understanding by means of exercises that use rods of increasing length was later adopted by Maria Montessori in [Montessori 1909, citation from 1912, 327].

Mentioned here are only a few of the methodologies, residing in studies of early education, which served to inform efforts to reform the teaching of Euclid.⁸ At the opposite end of the spectrum were proposals that were largely infused by research at the university level.

3 Rational intuition as an impetus for pedagogical reform: Demonstrative geometry

“Demonstrative geometry” has been described as the teaching of geometry with an emphasis on logical reasoning. It pays particular attention to rigor in remedying the logical lapses of the *Elements*. By the early 1900s, it enjoyed a prominent place in the United States curriculum. The American

7. Spencer's son, Herbert, a noted educator and philosopher, republished the book and wrote an introduction. Many editions were adopted as textbooks to transition students gradually from concrete thinking to abstract thinking.

8. For example, Enriques referenced Fröbel in [Enriques 1900, 26]. Klein weighed in on Pestalozzi [Lorenat 2020, 81]. See [Giacardi 2012], [Roberts 2014], [Millán Gasca 2015].

mathematician, David Eugene Smith, considered one of the founders of the field of mathematics education, proposed that “as an introduction to mathematics, it is the right and privilege of every student to know what demonstrative geometry means since that is where many students first awaken to the significance of mathematics” [Smith 1919, 112–113]. Those who advocated on behalf of demonstrative geometry cited, among its benefits, the power in uniquely developing the habit of deductive thinking [Hart 1924, 172].

At the same time in Italy, the intense production of technical papers, expository essays, textbooks, and publications addressed to mathematics teachers revealed deep connections between logical research and pedagogy, see [Giacardi & Scoth 2014]. The Peano School at the University of Turin flourished as a center for research in analysis, logic, foundations, and teaching. Peano and members of his School championed the role of logic to achieve more rigor in geometry. Prominent among them was Pieri—who explicitly recognized the “pedagogical and didactic” power of purely logical methods, attributing to them “the only capability to expose known truths”, with an “economy of labor” as compared with inductive reasoning from experience [Pieri 1906, 56–57(1980, 442–443)]. Yet, while Pieri’s immersion in the Peano School at the University of Turin is primarily seen as the context in which to view his foundational works, it is important to recognize also the influence of Pieri’s long and enduring membership in Segre’s School of algebraic geometry there. Aldo Brigaglia has demonstrated how Pieri’s research in the field of algebraic geometry, under the mentorship of Segre, did not conflict with his axiomatic research and in fact closely intersected with it, see [Brigaglia 2012].⁹ Referencing Pieri’s first axiomatization of projective geometry (three Notes published between 1894 and 1896), Brigaglia observed that Pieri made use of “the *teachings* of Peano to bring to fruition a scientific program developed by Segre” [Brigaglia 2012, 26].

Pieri’s proposals for teaching geometry are rooted in the symbiotic relationship between axiomatics and pedagogy. It was almost exclusively in his axiomatizations of elementary geometry [Pieri 1900a, 1908] that he explicitly talked about teaching. He did address the idea of geometric intuition in earlier foundational works. For example, to underscore his intention to develop projective geometry with a *puramente deduttivo ed astratto* approach, Pieri explained that by *abstract*, he meant,

[...] it disregards any *physical* interpretation of the premises, and therefore also their *evidence*, and *geometric intuitiveness*: unlike another direction (which I would call *physical-geometric*) according to which primitive entities and axioms want to be inferred from direct observation of the external world, and identified with the ideas that are acquired through experimental induction from

9. Brigaglia included a deep discussion of rigor in the application of the axiomatic method that informs the research of Pieri vis-à-vis his mentors.

certain physical objects and facts (PASCH, PEANO ...). [Pieri 1896, 10(1980, 84)]

In his constructions of *abstract* geometry, primitive notions, arbitrarily chosen, are implicitly defined by the postulates,¹⁰ and theorems are formally derived using Peano's mathematical logic. Geometry, so developed as a *hypothetical deductive* system, distinguishes itself from the classical *content-centered* Euclidean system where primitive concepts are based on verifiable evidence that is "transmitted to derived propositions through the definition of evidence and the demonstration of veracity" [Ingaliso 2011, 242].

Envisioning geometry as a hypothetical deductive science was, in Pieri's view, the optimal way to teach it. It would foster, in students, *rational* intuition, which he defined as "a perception of the logical relationship between principles and consequences" [Pieri 1906, 58(1980, 444)]. Pieri saw great benefits for students to develop such an intuition because,

a mind educated with general ideas and supported by a reasonable faculty of abstraction becomes capable of perceiving also, beyond the abstract logical sense, the connection between the various propositions and their deductive roles, the concatenation of the parts and their relationships with the whole. [Pieri 1908, 447(1980, 557)]

Pieri emphasized that *rational* intuition is "purely logical and intellectual syntheses, which do not admit anything from *sensible* intuition" [Pieri 1906, 58(1980, 444)]. However, he *only* excluded *sensible* intuition as an instrument to justify deductive reasoning [Pieri 1906, 35(1980, 421)]. In this he agreed with Moritz Pasch¹¹ who viewed such a use as a "sign of a shortcoming of deductions" [Pasch 1882, 82].

Pieri believed that geometry "as a formal science, should [...] be able to stand and to be understood without ever appealing to [...] intuitive or physical representations".¹² However, he explicitly noted "the heuristic importance, and [...] the didactic value, of [...] concrete interpretation[s] of geometric entities" [Pieri 1906, 47(1980), 413]. As early as 1889, in his annotated translation of [von Staudt 1847], Pieri had advocated for concrete interpretations in works "of geometry aimed especially at youth". He proposed:

10. Pieri's understanding of primitives and postulates reflected his selective endorsement of views expressed by Peano and Segre, see [Bottazzini 2001].

11. Pasch gave two "conditions" to define the concept of rigor that eschewed intuition [Pasch 1882, 16].

12. Pieri spoke eloquently about intuition in an address [Pieri 1906, 35(1980, 421)] delivered at the University of Catania, see [Ingaliso 2011], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §9.4.3]. For a discussion of rigor and axiomatics, see [Israel 1981].

Only after having constructed the figure of a demonstration, by himself and without any preconceptions, do we believe that the reader can assume that he has mastered this. [von Staudt 1889, xxv]

The important message here is that while Pieri advocated abstraction in the deductive process of proof, he saw the need for the visualization of a concrete image and/or construction of a figure to claim comprehension of that process and its outcome.¹³

We extract from these and other comments in his papers, our understanding of Pieri's stance on the pedagogical role of intuition. It is the responsibility of the teacher to promote in students a *rational* intuition of geometry as an abstract science. Among the benefits for students is that they learn how to draw consequences from principles, using logic, in a way that enables them to understand these consequences in relation not only to the principles, but also to one another. Students are encouraged to determine if these consequences conform to their *sensible* intuition of them, not only to assess their comprehension of them, but after deducing them logically, to appreciate the process used to derive them.

The underlying rationale of Pieri's proposal for educational reform thus emerges. He sees no contradiction in promoting both the intuitive and logical aspects of geometry in its teaching.

4 Pieri's strategies for teaching demonstrative geometry: Hypothetical-deductive systems and transformations

Pieri saw mathematics developed as a hypothetical-deductive system, not only as a means to achieve rigor and precision, but also as an opportunity to simplify its teaching in ways that promote in students a deeper understanding and appreciation of the subject. He proposed that the combination of the abstract with a transformational approach, in an exposition which simultaneously develops two and three-dimensional geometry, would make the subject more accessible to students.

Pieri believed he had experienced a level of success in his quest for simplicity with respect to his axiomatizations of elementary geometry. He

13. Observing that Pieri "had proposed construction as a yardstick for assessing geometric mastery of a proof", Lorenat characterized Pieri's "claim for learning geometry" as representing a "crucial" distinction between him and Pasch [Lorenat 2020, 82].

claimed that in his exposition of absolute¹⁴ geometry, he had achieved “such a degree of *deductive simplicity* that educational systems can certainly also take advantage of it” [Pieri 1900a, 3(1980, 185)]. Still, he called for further research to effect an even greater simplification. Answering his own call several years later with an axiomatization of Euclidean geometry [Pieri 1908], Pieri wrote: “Born of such research is the present Essay, which (if I am not mistaken) reaches exactly that degree of simplicity and rigor that I had in mind at that time, and which to my eyes represents the maximum value of this type of investigation” [Pieri 1908, 345(1980, 455)].

What did Pieri mean by *deductive simplicity*? This question is difficult to answer because although Pieri repeatedly emphasized its importance, he did not provide a formal definition.

Victor Pambuccian has systematically explored the many different ways to define the simplicity of an axiom system, see [Pambuccian 1988]. Under the rubric of purely syntactic considerations, one criterion addresses the language in which the axiom system is expressed. In this context “simple language” means having primitives that are both few in number and of lowest possible arity. Others concern the axioms themselves, asking for the fewest number of quantifier alternations appearing in each axiom or seeking a minimization of the number of variables appearing in axioms containing the largest number of variables [Pambuccian 2009, 328]. Another class of simple axiomatizations is the quantifier free one, in languages that contain only operation symbols (but no predicate symbols), see [Pambuccian 2008]. Showing the independence of the axioms in a finite set has been stipulated as a requirement for simplicity, see [Mancosu, Zach *et al.* 2009, § 1.3]. For didactic considerations, axioms have been described as “simpler”, when they are “more intuitive” [Lolli 2011, § 4.4].

We can perhaps use some of these measures as a lens through which to examine Pieri's claims for simplicity. With respect to Pambuccian's descriptions of criteria for syntactic simplicity, we note that Pieri relied almost exclusively on principles expressed by first-order sentences in both [Pieri 1900a, 1908]. In this, he distanced himself from Peano who, for example in [Peano 1889], did not distinguish first-order from second-order quantification. First-order logic would only emerge as a coherent framework for logical studies in the 1920s. In the evolution of thought related to this, what Pieri accomplished received little explicit attention until Alfred Tarski brought [Pieri 1908] into the discussion, beginning with an address to a 1927 mathematical congress in Lwów, see [Pambuccian 2002], [Marchisotto & Smith 2007, § 5.2], [Smith 2010],[Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 10.4]. In more recent times, Pambuccian credited Pieri with accomplishing in [Pieri 1908] “the task of achieving the upmost simplicity of the language of Euclidean geometry” [Pambuccian 2009, 328].

14. This axiomatization focuses on Euclidean geometry as taught then in elementary courses, except for the theorems dependent on the Euclidean parallel postulate, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §§ 8, 9.3].

Pieri also addressed another of Pambuccian's simplicity criteria in his advocacy of the efforts of the Peano School to minimize the number of primitives in axiom systems. In both [Pieri 1900a, 1908], he reduced this number to two, respectively "point and motion", and "point and sphere (a single ternary equidistance relation)". When they recommended [Pieri 1900a] for publication, Enrico D'Ovidio and Segre observed,

This is a most notable result; and it does not seem that others previously have achieved such simplicity [...] From the purely logical point of view the system of Pieri is fully satisfactory, and contains [...] a result of particular importance in the reduction made in the primitive notions. [D'Ovidio & Segre 1899, 761]

Although Pieri believed that the independence of the postulates is a condition that nearly approaches "ideal perfection" [Pieri 1901, 380(1980, 248)], he did not prove the independence of his postulates for elementary geometry. Nonetheless we can say that he spoke to the criterion for simplicity specified in Mancosu, Zach *et al.* [2009] when he proved the ordinal independence of his [Pieri 1898] postulates for projective geometry.

With respect to Lolli's criteria for simplicity, we note Pieri's deep-rooted belief that the hypothetical-deductive approach would lead students to new questions and deeper intuitions of the subject [Pieri 1900a, 177(1980, 187)]. Consider, for example, his strategy for developing the notion of line in [Pieri 1900a]. He began with a postulate stated solely in terms of the primitives, point and motion,

For distinct points A , B , C : If there exists a non-identity motion that fixes A and B , it will also fix C . [Pieri 1900a, § 1 Postulate VIII, 182(1980, 192)]

Pieri remarked, "This is a principle of great deductive capacity [...]. It is now given to us to produce and develop through it the notion of 'line' and to recognize some of its more notable properties." Observing that postulate VIII (and his definition of collinear points based upon it) cannot describe the geometry of hyperspace, Pieri explained that he chose to consider "only elementary geometry, seeking as much as possible to establish the principles in a manner more suitable to deductive simplicity" [Pieri 1900a, 182(1980, 192)]. To develop the notion of line, Pieri followed his usual practice of formulating definitions explicitly (in terms of primitives) or implicitly (in terms of postulates). He used postulate VIII to define collinearity in terms of motion: a point is collinear with two given points if there exists a non-identity motion that fixes all three points. A straight line on two given points is then the line of points collinear with them, a unique line that remains motionless when it rotates around those two points, see [Marchisotto 1992].

Ugo Amaldi observed that with [Pieri 1900a], Pieri had established "a rigorous logical structure" based on this definition of line [Amaldi 1912-1914, 42]. Amaldi's use of the term "rigorous" would be, for Pieri, an affirmation of

the simplicity he sought in developing geometry as “a study of a certain order of logical relations” [Pieri 1908, Preface 347(1980, 457)]. Indeed to that very description in [Pieri 1908], Pieri (in a footnote referencing [Halsted 1904, 189]) included a quotation from [Hilbert 1900] proposing that the most rigorous method is often the simplest and the easiest to comprehend.

These examples suggest that Pieri saw the rigorous application of axiomatic method as one which enabled an exposition of geometry that he could characterize as *deductively simple*.¹⁵ To that end, using a minimal number of primitives, he “unfolded” geometry axiomatically, introducing postulates and definitions only as needed to derive theorems by applying the laws of logic. His proofs were rigorously executed and extraordinarily detailed, showing precisely which postulates, definitions, and previously-proved theorems justified his statements. In these ways, Pieri hoped to promote in students a mindfulness of the deductive role that collections of postulates played with respect to when and how theorems are proved. He went to great lengths to encourage deep thinking about geometric propositions, exposing how postulates, definitions and theorems relate to each other in relation to the geometric edifice being constructed.

Still, Pieri's quest for simplicity was not restricted to logical deduction. His vision for teaching geometry as an abstract science emerged not only from his embrace of deductive logic, but also from his pioneering use of Klein's transformational approach to geometry, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §§ 10.2, 10.4]. He believed the integration of these elements could address the challenges of “reconciling the needs of schools with the ideals of the deductive method”. In his review of a secondary school textbook written by Giuseppe Ingrams [Ingrams 1899], Pieri described such a reconciliation as an “undertaking if it ever comes to be”, which will be a result of “the toil and exertion of many” [Pieri 1899, 181].

For Pieri, the transformational approach would facilitate the teaching of geometry as a hypothetical-deductive system in a way that encouraged students to appreciate the duality between the abstract nature of mathematical objects and their concrete representations. He noted that,

The act of using the simplest motions, such as translations, rotations, symmetries, and so on, and their products, more broadly than usual in definitions and arguments confers on the system as a whole a certain ease of manner that is not devoid of clarity and effectiveness. [Pieri 1901, 384 (1980, 252)]

15. Pambuccian (in a personal communication of June 2020) has suggested that since Pieri referenced deduction, his idea of simplicity can be interpreted in the arena of what is today known as Hilbert's “24th Problem”. See [Pambuccian 2019], which analyzes the manner in which the simplicity of proofs could be defined using concrete examples from other works of elementary geometry.

In his axiomatization of absolute geometry [Pieri 1900a], Pieri followed [Peano 1894] in choosing direct motion as primitive.¹⁶ The Italian philosopher Antonio Aliotta called this choice “intuitive” [Aliotta 1911, citation from 1914, 317]. Bertrand Russell noted how Pieri used it¹⁷ to its best advantage,

Pieri has shown, in an admirable memoir, how to deduce metrical geometry by taking point and motion as the only indefinables. [Earlier] we objected to the introduction of motion, as usually effected, on the ground that its definition presupposes metrical properties; but Pieri escapes this objection by not defining motion at all, except through the postulates [...]. The straight line is the class of points that are unchanged by a motion that leaves the two points fixed. The sphere, the plane, perpendicularity, order of points on a line, etc., are easily defined. This procedure is logically unimpeachable, and is probably the simplest possible for elementary Geometry. [Russell 1903, §395]

Consider, for example, Pieri’s treatment of perpendicularity in [Pieri 1900a, §2P19, 190–191 (1980, 200–201)]. He “easily” defined it in terms of the existence of a motion,

Let A, B, C be points such that A is different from B and C . We say AC is perpendicular to AB if and only if there exists a motion that leaves A and B invariant while transforming C into a point of AC different from C .

He next gave two ways to interpret this definition,

Given non-collinear points A, B, C , in the rotation of their plane onto itself about A and B as hinges, C falls back onto the line AC .

There exists a proper (non-identity) motion that leaves the points A and B fixed, bringing the line AC back onto itself.

After which, illustrating the melding of its transformational and abstract aspects, he set the discussion in the context of the scheme of the logical-deductive plan,

Here orthogonality is introduced in the form of a relation among three given points and no other, thus restored to its primitive terms and divested of all that is superfluous (with respect to our system). Thus, it is in the nature of algebraic logic.

16. For a broader discussion of the commonalities and differences between Peano and Pieri relative to their treatment of geometry and its transformations, see [Marchisotto 2011].

17. Despite the fact that no direct influence has been shown so far, Pieri’s approach has proved to be extraordinarily fruitful. In particular, Johannes Hjelmslev’s approach, continued by Arnold Schmidt, and finally by Friedrich Bachmann, created an axiomatization of absolute geometry based entirely on “motions”, see [Pambuccian, Struve *et al.* 2017], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §7.5].

Thus, Pieri saw the integration of abstract and transformational approaches as a compelling pedagogical strategy. To further enhance its effectiveness, he appealed, albeit not explicitly, to fusionism.¹⁸ Riccardo de Paolis had proposed this method, which treats plane and solid geometry simultaneously, to improve its teaching. He noted, “[...] there exist many analogies between certain figures of the plane and certain figures of space” for which reason “in studying them separately we lose the knowledge of everything these analogies teach us and voluntarily fall into useless repetition” [De Paolis 1884, Appendix 2].

Cremona had suggested that stereo-metric considerations often suggest a way to simplify complicated proofs in plane geometry and make them more intuitive.¹⁹ He recommended alternating theorems of plane and solid geometry in the high school teaching of projective geometry to sharpen the intellect of students and help them develop “geometric imagination” [Cremona 1873, xi]. Yet traditions in teaching and other methodological commitments (related to “purity of methods”) impeded the adoption of such strategies for Euclidean geometry [Arana & Mancosu 2012, 303].

Pieri, however, was not dissuaded. Fusion served his didactic objectives. It helped him show, among other things, that a geometric property does not exist in isolation. Pieri believed that when a property

appears at the same time in several hypotheses and in different propositions, [...] it comes to connect with other properties: then only the logical process of deduction is sufficient to recognize in these other properties the existence of new bonds and new connections. [Pieri 1906, 42(1980, 408)]

For example, in [Pieri 1900a, §3P20, 197-98(1980, 207-08)], Pieri demonstrated how one property can be used to define the reflection point B of a point A across a line r or across a plane Π : that the point for which the midpoint of A and B is the foot of the perpendicular from A to Π . It was Pieri's practice to append to his formal definitions, comments intended to clarify their meaning. Such observations here appended to P20 demonstrate the confluence of his transformational and fusionistic approaches,

To say that points A and B are symmetric to each other with respect to a line r or with respect to a plane Π will be like asserting that these points are both on a line that meets r or Π orthogonally at the midpoint of AB .

By means of a line or a plane, then, there is determined a certain representation of points by points (of space into itself),

18. See [Arana & Mancosu 2012] for a discussion that includes an exposition of historical, methodological and foundational aspects of fusionism.

19. When he was in Rome, Cremona studied with De Paolis. After reading [Pasch 1882], he sent to its author a copy of [De Paolis 1880-1881] on the foundations of projective geometry, see [Millán Gasca 2017].

such that each point A of r or of Π is associated with the same point B .

This transformation or geometric correspondence is to be called axial symmetry with respect to r in the one case, and planar symmetry with respect to Π in the other.

Pieri next focused attention on the abstract deductive scheme by advocating the consultation of previous propositions (§1P11, §1P20 and §3P1) for an understanding of why a planar symmetry cannot be a direct motion (all the points of a certain plane Π are fixed there, but not all the points that exist). He then showed that any arbitrary axial symmetry is a direct motion, which exists by virtue of §3P20. Pieri proved that given three noncollinear points, any motion that fixes two of them and transforms the third into its symmetric point with respect to the line joining those two points will also have to change any other point into its symmetric point with respect to that line. His proof of this statement (summarized here) illustrates the level of detail in his exposition, constructed to reveal precisely how deduction logically proceeds from definitions and previously proven propositions:

Given noncollinear points A, B, C , let μ be a motion that fixes A and B and transforms C into its symmetric point with respect to AB . By §1P5, §1P20, §1P22, §1P24, §2P17, etc., μ^2 is a motion that fixes A, B , and C , hence by §1P11 leaves every point invariant. Therefore, if $\mu(Z) = Z'$, the points Z and Z' are permuted with each other by μ . Thus, by §2P5-P7, etc., their midpoint X is fixed and thus will have to lie on AB , since μ , by hypothesis, is a (non-identity) motion. It follows by §2P19, §2P27, etc., that either $Z = Z' = X$, or XZ is perpendicular to AB . Thus in each case, Z' is the point symmetric to Z with respect to AB , and μ is none other than the rotation of the plane ABC onto itself about A and B as hinges (see §2P11), etc. [Pieri 1900a, §3P21, 198(1980, 208)]

We have provided these few examples to convey the essence of Pieri's proposals for reforming the teaching of Euclidean geometry and to illustrate how his views on intuition informed them. We next address how Pieri's ideas were interpreted by several of his contemporaries.

5 A chorus of voices: the reception of Pieri's pedagogical views

Pieri made his proposals for reform at a time when there was considerable support for the idea that research in foundations could help transform pedagogy. Those who endorsed building on the connections between foundations of

mathematics and its teaching debated about the paths to do so, the underlying epistemological ideas, the rudiments of student understanding, and more—producing a *chorus* of voices, interpretations, and projects. We show how Pieri was represented in three influential publications in the early decades of the 20th century that served as forums for this *chorus*.

We begin with the prestigious *Encyklopädie der mathematischen Wissenschaften*²⁰ [Meyer & Mohrmann 1907-1934], and the article, entitled “Prinzipien der Geometrie” written by Federigo Enriques [Enriques 1907]. In his reference to [Pieri 1900a, 173], where Pieri had described what he meant by hypothetical-deductive system, Enriques observed that “*M. Pieri* defined ‘segment’ using the concepts ‘point’ and ‘motion’” and, to that end, “had developed a system of postulates”. He continued,

It should be noted that so far *Pieri* alone has formulated in a complete way the postulates. However, these postulates, mainly due to the fact that the primitive concepts of order (i.e., the attributes of the straight line regarding only its being a line) were deleted, come in an extremely complicated form and lose all clarity and intuitive certainty (*[sic]* anschauliche Gewißheit); however Pieri attaches no importance to this feature. [Enriques 1907, citation from 1907-1910 § 6, 33]

Enriques’ assertion that Pieri “attaches no importance” to the “intuitive certainty” of his postulates is a misrepresentation. Indeed a year before Enrique’s article appeared, in an address for the inauguration of academic year at the University of Catania, Pieri explicitly warned against denying the role of “ingenious intuition”, in the logical process [Pieri 1906, 79–80(1980, 445–446)].

In his revision for the 1911 French edition of the encyclopedia, Enriques focused his comments about Pieri’s postulates more directly on the notion of “evidence”, saying:

It should be noted that so far *M. Pieri* alone has formulated in a *complete* way, the postulates of which he makes use. It should be added, however, that these postulates come in an extremely complicated form and lose all obviousness (*[sic]* tout caractère d’évidence) relative to the intuition we have of them. This is mainly due to the fact that the primitive concepts of order (that is to say, the attributes of the line as a line) were removed by *M. Pieri*. Besides, *M. Pieri* attaches no importance to the greater

20. The first volume of the first edition of this German encyclopedia appeared in 1899. The original project, initiated by Klein, Heinrich Weber and Franz Meyer, sought to compile and present a comprehensive review of the science of mathematics and its allied fields. It was considered a monumental and ambitious task which aroused great interest among contemporary mathematicians [Ore 1942, 653].

or lesser *evidence* of his premises. [Enriques 1911-1915, citation from 1911, §12, 33]

It is likely that Enriques' italicized reference to "evidence" spoke to his belief that the postulates of geometry are just rigorous forms of the intuitive concept of physical space, and his recognition that this was a belief to which Pieri did not subscribe. However, saying that Pieri attached "no importance" to the evidence of his premises is an inaccurate characterization. Although he believed that the *source* of primitive ideas and postulates resides "in the domain of abstractions", Pieri also stressed that they "must find an image [...] exact and in accordance, if not perfect agreement, with every sort of objects and phenomena to which one would apply the system in whole or in part" [Pieri 1901, 379(1980, 247)].

Amaldi was more accurate in conveying Pieri's stance on intuition. Amaldi had collaborated with Enriques on a noted anthology [Enriques 1900] that had been compiled as a resource for making changes in higher mathematics teaching for the preparation of secondary school teachers.²¹ While he did not cite Pieri in his essay for the first edition, "Sui concetti di retta e di piano" [Amaldi 1900, 33-64], he did reference [Pieri 1898, 1900a, 1908] in the expanded version of that essay for the second edition [Amaldi 1912-1914, 37-91]. Unlike Enriques, Amaldi understood and reported that Pieri made no appeal to intuition in *formulating* his postulates:

Starting from the formal point of view of reducing the number of primitive ideas [...] leaving aside any other need for the postulates with respect to intuition, [...] first analyzing the principles of projective geometry and then those of elementary geometry, he showed that all geometry can be built with only two primitive ideas, those of *point* and *distance*. [Amaldi 1912-1914, 79]

Thus, Amaldi recognized the fact that Pieri excluded the consideration of intuition in composing his primitives (and the postulates that define them). However, Amaldi neglected to convey Pieri's views, from both foundational and pedagogical perspectives, on the role of intuition in interpreting them. Therefore, Amaldi had told only part of the story. Pieri had emphasized,

Learning geometric facts is greatly helped by always having at the onset an image or intuitive representation of a "point" and of the "sphere through one point centered at another": that is, the habit of contemplating the real and concrete sense that usage provides for statements such as "*A*, *B*, *C* are points, and *C* is as distant from *A* as *B* is". [Pieri 1908, 447(1980, 557)]

21. The spirit that led to the publication of the first edition was pervasive in Italy and beyond. For example, Julio Rey Pastor, who believed Enriques was among those mainly responsible for the influx of foundational research in secondary education, brought this same spirit to Spain and Latin America, after spending time in Italy in 1914, see [Millán Gasca 1990], [Giacardi 2012].

Pieri's words here call to mind those of a fellow member of the Peano School, Giovanni Vailati, who characterized [Pieri 1908] as a "step forward" in treating the subject from "*the most general possible viewpoint—compatible with the concrete material to which it refers*" [Arrighi 1997, letter 126, June 25, 1908]. Amaldi appeared to agree, observing that [Pieri 1908] provided "a complete analysis of the foundations of Euclidean geometry", and using the words "perspicacious" and "simple" to characterize Pieri's system [Amaldi 1912-1914, 79]. Unfortunately, Amaldi's comments were removed in later editions of the anthology, while Enriques' mischaracterizations continued to be widely circulated for decades in the many editions and translations of the encyclopedia.

Enriques and Amaldi were prolific in the area of pedagogical research, authoring textbooks and publications devoted to teaching and the training of teachers. They were strong voices in the *chorus*, seeking to promote insights drawn from recent foundational, logical, psychological and historical research to improve the teaching of elementary geometry, while remaining faithful to Euclid. There is little reason to believe that their characterizations of Pieri would have motivated those seeking improvements in pedagogy in the early decades of the 20th century to examine Pieri's work. In the United States, a widely circulated didactic publication may have done a similar disservice to Pieri, in the context of its reporting on the research of the Peano School.

In [Cajori 1910], Florian Cajori, a Swiss-American historian and mathematician published a survey of worldwide pedagogical practices. Drawing from Klein, who used the word "radical" in his references to the Peano School [Klein 1909, citation from 2016, 262], Cajori reported,

A very remarkable school came into being in Italy, the purpose of which is to render geometry still more rigorous than in the Euclidean text. [...] the great school of Peano, which endeavors to eliminate all intuition [...] has influenced even elementary instruction and the teaching in technical schools. This recent Italian emphasis upon extreme rigor has led to deplorable results with the less gifted pupils, and a reaction appears to be setting in. [Cajori 1910, 192]

The charge of endeavoring to "eliminate all intuition" is certainly not one that applies to Pieri, nor, in fact, to others in the Peano School, see [Luciano 2012*b*].

Cajori's survey was reprinted in a 1912 Report issued by a Committee that had been formed under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences.²² Cajori's negative characterization of the Peano School thus

22. This "National Committee of Fifteen on Geometry Syllabus" was composed of representatives from universities and secondary schools. Although it had not completed its work in 1910, this committee advocated the early publication of Cajori's article, noting: "this historical setting prepared by Professor Cajori should be in the

had the potential to dissuade many from considering the pedagogical proposals of its members.

6 Pieri, in conversation with Klein

Klein was a significant influence for the works we cited in section 5. He was very much a hero to Pieri although their relationship was complex, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §10]. In this section, we reference an 1897 exchange of letters between Klein and Pieri. The words of the two mathematicians are particularly pertinent to our discussion about Pieri's pedagogy.

David E. Rowe observed that “Klein, like Poincaré, saw the burgeoning interest in abstract structures and axiomatics around the turn of the century as a potential threat to the lifeblood of mathematics”, but that being said, one “would be mistaken to think that Klein had no appreciation for axiomatic thinking” [Rowe 1989, 198]. A letter of March 31, 1897 reveals that Klein had engaged with Pieri in a discussion of axiomatics and intuition, relative to teaching [Arrighi 1997, Letter 65].

Referring to the “purely deductive mode of representation”, which he acknowledged “to be scientifically very important”, Klein had asked Pieri “what significance is to be given to this in teaching beginners”. Klein advised, “in teaching it is necessary to begin with intuition (in order then to gradually ascend to more abstract views)”. In his April 9, 1897 response to Klein, Pieri wrote,

To answer your question about the influence that these purely deductive research can have over the elementary teaching of mathematics (where intuition must have a most essential part) I will tell you that, in my opinion, many improvements in the strictly deductive sense, [...] would perhaps help to make these doctrines easier to understand than they are at present: since the greater abstraction would be compensated by the greater simplicity of the fundamental concepts. [Klein papers, Staats- und Universitätsbibliothek, Göttingen, 22F, 97–99]

In this exchange, we see once again that Pieri fully acknowledged the importance of intuition in the elementary teaching of mathematics. We also see, in the presence of this acknowledgement, Pieri's rationale for advocating an abstract approach.

Still Pieri could find no place for his pedagogical strategies, even among those like J. W. Young who admired his foundational works, see [Marchisotto & hands of mathematical teachers at once”. The survey was reprinted in their Final Report, published in 1912 in [National Committee of Fifteen on Geometry Syllabus 1912].

Smith 2007, 135, 142–43]. In [Young 1911, 164], Young indicated he could not recommend Pieri's assumptions for use in the school because "the subject must be presented to a boy of fourteen years in a different way from that employed in presenting it to a mature mind [...]", it being "necessary, in the beginning, to make continued and insistent appeal to concrete geometric intuition". This was precisely Klein's view. But it does not contradict what Pieri himself suggested. For the teaching of geometry, Pieri wrote,

[...] it will never be superfluous to appeal [...] to [...] empirical methods to emphasize and bring alive for the young all sorts of intuitive and experimental cognitions of various geometric objects. [Pieri 1908, 447, Note 2(1980, 557)]

Young had advocated starting with an informal treatment of geometry so that "the pupil could be led to see the advantages of the more formal methods that follow" [Young 1911, 163–164]. In our view, it is not clear why that formal treatment could not be along the lines of that proposed by Pieri. Indeed, Louis Couturat called [Pieri 1900a] "the most profound analysis of the principles of geometry" [Couturat 1905, 193].

7 Concluding remarks

Gino Loria, Pieri's colleague at the University of Turin, applauded Pieri's efforts to improve pedagogy. He called [Pieri 1900a] "a notable result, not just because of its simplicity and originality, but also because it seems directed toward those schools which would banish motion from pure geometry". Loria added, "The future will decide whether Pieri's ideas can lead to a useful reform of elementary instruction. What is without question, however, is that they deserve the attention of scholars and teachers" [Loria 1899, 426].

That did not happen. Pieri's proposals remained largely hidden from view. Enriques' publications with Amaldi, like those of Klein, which inspired them, had a great impact on pedagogical reform in the early decades of the 20th century. Cajori's historical survey of pedagogical practices enjoyed wide circulation. Pieri's ideas were absent or misstated in them. Sadly, the lack of dissemination and misrepresentation of Pieri's ideas are also evident in examining Peano's publication, the *Formulario*, see [Luciano 2017].

Pieri's pedagogical proposals did not garner attention in the transnational reform efforts of *his* era. For scholars and teachers of *this* era, we have endeavored to shed light on Pieri's ideas about how to integrate two types of intuition, denoted as *sensible* and *rational*, in efforts to improve the teaching of the geometry of Euclid. We have also wanted to reveal how certain representations of Pieri's ideas in the decade prior to his early death at age 52 in 1913, may have contributed to the obfuscation of what he truly proposed for such reform.

It is our hope that we have provided reasons why Pieri's ideas should have been heard in that *chorus* of voices, interpretations, and proposals being discussed at the turn to the 20th century. We trust we have also conveyed why today's mathematicians and mathematics educators would benefit from hearing his voice.

Acknowledgements

In addition to expressing our appreciation to those authors whose works assisted us in the preparation of this article, we would especially like to thank Paolo Mancosu and Victor Pambuccian for giving their time so generously to make constructive and valuable suggestions.

Bibliography

- ALIOTTA, Antonio [1911], *La reazione idealistica contro la scienza*, Palermo: Optima, cited according to the English translation by A. Mccaskil, *The Idealistic Reaction against Science*, London: Macmillan, 1914.
- AMALDI, Ugo [1900], Sui concetti di retta e di piano, in: *Questioni riguardanti la geometria elementari*, edited by F. Enriques, Bologna: Zanichelli, 33–64.
- [1912-1914], Sui concetto di retta e di piano, in: *Questioni riguardanti le matematiche elementari*, edited by F. Enriques, Bologna: Zanichelli, 37–91.
- ARANA, Andrew & MANCOSU, Paolo [2012], On the relationship between plane and solid geometry, *Review of Symbolic Logic*, 5(2), 294–353, doi: 10.1017/s1755020312000020.
- ARRIGHI, Gino [1997], *Lettere a Mario Pieri (1884-1913)*, *Quaderni P.RI.ST.EM.*, vol. 6, ELEUSI-Sezione P.RI.ST.EM., per l'archivio della corrispondenza dei matematici italiani.
- BARBIN, Evelyne & MENGHINI, Marta [2014], History of teaching geometry, in: *Handbook on the History of Mathematics Education*, edited by A. Karp & G. Schubring, New York: Springer, 473–492, doi: 10.1007/978-1-4614-9155-2_23.
- BETTI, Enrico & BRIOSCHI, Francesco (eds.) [1867], *Gli Elementi d'Euclide*, Florence: Le Monnier.
- BETZ, William [1933], The teaching of intuitive geometry, in: *The Teaching of Mathematics in the Secondary School, National Council of Teachers of Mathematics Yearbook*, edited by W. D. Reeve, New York: Columbia University, 8th edn., 55–164.

- BOOLE, Mary Everest [1904], *The Preparation of the Child for Science*, Oxford: Clarendon Press.
- BOTTAZZINI, Umberto [2001], I geometri italiani e i *Grundlagen der Geometrie* di Hilbert, *Bollettino dell'Unione Matematica Italiana*, 4-B(3), 545–570, URL <http://eudml.org/doc/195252>.
- BRIGAGLIA, Aldo [2012], Mario Pieri e la Scuola di Corrado Segre, in: *Quaderni di Storia della Università di Torino*, edited by C. S. Roero, Turin: Celid, 19–34.
- CAJORI, Florian [1910], Attempts made during the eighteenth and nineteenth centuries to reform the teaching of geometry, *The American Mathematical Monthly*, 17(10), 181–201, doi: 10.2307/2973645.
- CLAIRAUT, Alexis [1741-1852], *Éléments de géométrie*, Paris: David Fils.
- COUTURAT, Louis [1905], *Les Principes des mathématiques*, Paris: Alcan.
- CREMONA, Luigi [1873], *Elementi di geometria proiettiva*, Rome: G. B. Paravia.
- DE PAOLIS, Riccardo [1880-1881], Sui fondamenti della geometria proiettiva, in: *Atti della Reale Accademia dei Lincei: Memorie della Classe di Scienze Fisiche, Matematiche e Naturali, Series 3*, vol. 9, 489–503.
- [1884], *Elementi di geometria*, Turin: Ermanno Loescher.
- D'ENFERT, Renaud [2003], *L'Enseignement du dessin en France. Figure humaine et dessin géométrique (1750-1850)*, Paris: Belin.
- D'OVIDIO, Enrico & SEGRE, Corrado [1899], Relazione sulla memoria del Prof. M. Pieri, intitolata: Della Geometria Elementare come sistema ipotetico-deduttivo, in: *Atti della Reale Accademia delle Scienze di Torino*, vol. 34, 760–762.
- ENRIQUES, Federigo (ed.) [1900], *Questioni riguardanti la geometria elementare*, Bologna: Zanichelli.
- ENRIQUES, Federigo [1907], Prinzipien der Geometrie, in: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, edited by F. Meyer & H. Mohrmann, Leipzig: B. G. Teubner, vol. 3: Geometrie, 1–129, cited from the [1907-1910 Edition].
- [1911-1915], Principes de la géométrie, in: *Encyclopédie des sciences mathématiques pures et appliquées*, edited by J. Molk & Fr. Meyer, Paris: Éditions Jacques Gabay, vol. 3: Géométrie, 1–147, cited from the [1911 Edition].

- FRÖBEL, Friedrich [1826], *Die Menschenerziehung*, Leipzig: Wienbrack.
- GIACARDI, Livia [2012], Federigo Enriques (1871-1946) and the training of mathematics teachers in Italy, in: *Mathematicians in Bologna 1861-1960*, edited by S. Coen, Basel: Springer, 209–275, doi: 10.1007/978-3-0348-0227-7_9.
- GIACARDI, Livia & SCOTH, Roberto [2014], Secondary school mathematics teaching from the early nineteenth century to the mid-twentieth century in Italy, in: *Handbook on the History of Mathematics Education*, edited by A. Karp & G. Schubring, New York: Springer, 201–228, doi: 10.1007/978-1-4614-9155-2_10.
- GIUSTI, Enrico [1993], *Euclides reformatus. La teoria delle proporzioni nella scuola galileiana*, Turin: Bollati Boringhieri.
- HALSTED, George B. [1904], The message of non-Euclidean geometry, *Science*, 19(480), 401–413, doi: 10.1126/science.19.480.401.
- HART, W. W. [1924], Purpose, method, and mode of demonstrative geometry, *The Mathematics Teacher*, 17(3), 170–177, doi: 10.2307/27950607.
- HENDRIX, Gertrude [1936], A protest against informal reasoning as an approach to demonstrative geometry, *The Mathematics Teacher MT*, 29(4), 178–180, doi: 10.2307/27951923.
- HILBERT, David [1900], Mathematische Probleme, *Nachrichten von der Königlichlichen Gesellschaft der Wissenschaften zu Göttingen*, 3, 253–297.
- INGALISO, Luigi [2011], Mario Pieri's address at the University of Catania, *Historia Mathematica*, 38(2), 232–247, doi: 10.1016/j.hm.2010.05.004.
- INGRAMI, Giuseppe [1899], *Elementi di geometria per le scuole secondary superiori*, Bologna: Tipografia Cenerelli.
- ISRAEL, Giorgio [1981], “Rigor” and “Axiomatics” in modern mathematics, *Fundamenta Scientiae*, 2(2), 205–219.
- [2017], Luigi Cremona, in: *Correspondence of Luigi Cremona (1830-1903), conserved in the Department of Mathematics, “Sapienza” Università di Roma*, Thurnout: Brepols, 33–58.
- KARP, Alexande & SCHUBRING, Gert (eds.) [2014], *Handbook on the History of Mathematics Education*, New York: Springer.
- KLEIN, Felix [1909], *Elementarmathematik vom höheren Standpunkte aus*, Leipzig: B. G. Teubner, doi: 10.1007/978-3-662-49442-4, cited according to the English translation by M. Menghini & G. Schubring: *Elementary Mathematics from a Higher Standpoint*, Berlin: Springer, 2016.

- LEGENDRE, Adrien-Marie [1752-1833], *Éléments de géométrie*, Paris: F. Didot, cited from the 1823 edition.
- LOLLI, Gabriele [2011], Peano and the foundations of arithmetic, in: *Giuseppe Peano between Mathematics and Logic*, edited by F. Skof, Milan: Springer, 47–67, doi: 10.1007/978-88-470-1836-5_4.
- LORENAT, Jemma [2020], Drawing on the imagination: The limits of illustrated figures in nineteenth-century geometry, *Studies in History and Philosophy of Science Part A*, 82, 75–87, doi: 10.1016/j.shpsa.2020.01.002.
- LORIA, Gino [1899], Reviews 30.04026.02, 30.0426.03, *Jahrbuch über die Fortschritte der Mathematik*, 30, 426–428.
- LUCIANO, Erika [2012a], Mario Pieri e la scuola di Giuseppe Peano, *Quaderni di Storia dell'Università di Torino*, 10 (2009-2011), 35–62.
- [2012b], The proposals of the School of Peano on the rational teaching of geometry, in: *Dig where you stand, 2. Proceedings of the Second International Conference on the History of Mathematics Education, October 2-5, 2011*, edited by K. Bjarnadóttir, F. Furinghetti, J. M. Matos, & G. Schubring, New University of Lisbon, Portugal, UIED, 281–302.
- [2017], Characterizing a mathematical school: Shared knowledge and Peano's *Formulario*, *Revue d'Histoire des mathématiques*, 23, 183–231.
- MANCOSU, Paolo, ZACH, Richard, *et al.* [2009], The development of mathematical logic from Russell to Tarski: 1900-1935, in: *The Development of Modern Logic*, edited by L. Haaparanta, New York: Oxford University Press, 318–370, doi: 10.1093/acprof:oso/9780195137316.003.0029.
- MARCHISOTTO, Elena Anne Corie [1992], Lines without order, *The American Mathematical Monthly*, 99(8), 738–745, doi: 10.2307/2324240.
- [2010], Mario Pieri: l'Uomo, il Matematico, il Docente, *La Matematica nella Società e nella Cultura Rivista dell'Unione Matematica Italiana*, III(Serie I), 321–364.
- [2011], Foundations of geometry in the school of Peano, in: *Giuseppe Peano between Mathematics and Logic*, edited by F. Skof, Milan: Springer, 157–169, doi: 10.1007/978-88-470-1836-5_9.
- MARCHISOTTO, Elena Anne Corie, RODRÍGUEZ-CONSUEGRA, Francisco, *et al.* [2020], *The Legacy of Mario Pieri on the Foundations and Philosophy of Mathematics*, Boston: Springer.
- MARCHISOTTO, Elena Anne Corie & SMITH, James T. [2007], *The Legacy of Mario Pieri in Geometry and Arithmetic*, Boston: Birkhäuser, doi: 10.1007/978-0-8176-4603-5.

- MENGHINI, Marta [2015], From practical geometry to the laboratory method: The search for an alternative to Euclid in the history of teaching geometry, in: *Selected Regular Lectures from the 12th International Congress on Mathematical Education*, edited by S. J. Cho, Cham: Springer, 561–587, doi: 10.1007/978-3-319-17187-6_32.
- MEYER, Franz & MOHRMANN, Hans (eds.) [1907-1934], *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, vol. 3: Geometrie, Leipzig: B. G. Teubner.
- MILLÁN GASCA, Ana [1990], Methods of synthetic geometry in the second half of the nineteenth century, in: *International Symposium Structures in Mathematical Theories (San Sebastián, 25–29 September 1990)*, edited by A. Díez, J. Echevarría, & A. Ibarra, San Sebastián: Universidad del País Vasco, 283–288.
- [2011], Mathematicians and the nation in the second half of the nineteenth century as reflected in the Luigi Cremona correspondence, *Science in Context*, 24(1), 43–72, doi: 10.1017/S0269889710000268.
- [2015], Mathematics and children’s minds: The role of geometry in the european tradition from Pestalozzi to Laisant, *Archives Internationales d’Histoire des Sciences*, 65(175), 759–775, doi: 10.1484/J.ARIHS.5.112785.
- [2017], Letter from Moritz Pasch (1882), in: *Correspondence of Luigi Cremona (1830-1903) conserved in the Department of Mathematics, “Sapienza” Università di Roma*, edited by G. Israel, Thuin: Brepols, vol. 97, 1287–1288.
- MINCHIN, George [1898], *Geometry for Beginners*, Oxford: Clarendon Press.
- MONTESSORI, Maria [1909], *Il metodo della pedagogia scientifica applicato all’educazione infantile nelle case dei bambini*, Rome: M. Bretschneider, cited according to the English translation by A. E. George, *The Montessori Method*, New York: Frederick A. Stokes Company, 2012.
- NATIONAL COMMITTEE OF FIFTEEN ON GEOMETRY SYLLABUS [1912], Final Report of the National Committee of Fifteen on Geometry Syllabus, *The Mathematics Teacher*, 5(2), 46–131, doi: 10.2307/27949764.
- ORE, Øystein [1942], Review: Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, *Bulletin of the American Mathematical Society*, 48(9), 653–658, doi: 10.1090/S0002-9904-1942-07730-5.
- OSBECK, Lisa M. & HELD, Barbara S. (eds.) [2014], *Rational Intuition: Philosophical roots, scientific investigations*, New York: Cambridge University Press.

- PAMBUCCIAN, Victor [1988], Simplicity, *Notre Dame Journal of Formal Logic*, 29(3), 396–411, doi: 10.1305/ndjfl/1093637936.
- [2002], Axiomatizations of hyperbolic geometry: A comparison based on language and quantifier type complexity, *Synthese*, 133(3), 331–341, doi: 10.1023/A:1021294808742, correction, *Synthese*, 145, 497: doi: 10.1007/s11229-004-5405-5.
- [2005], The complexity of plane hyperbolic incidence geometry is $\forall\exists\forall\exists$, *Mathematical Logic Quarterly*, 51(3), 277–281, doi: 10.1002/malq.200410028, Corrigendum: 54, 668.
- [2008], Axiomatizing geometric constructions, *Journal of Applied Logic*, 6(1), 24–46, doi: 10.1016/j.jal.2007.02.001.
- [2009], Universal-existential axiom systems for geometries expressed with Pieri's isosceles triangle as single primitive notion, in: *Rendiconti del Seminario Matematico, Università e Politecnico di Torino*, vol. 67, 327–339.
- [2019], Prolegomena to any theory of proof simplicity, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 377(2140), 20180 035, doi: 10.1098/rsta.2018.0035.
- PAMBUCCIAN, Victor, STRUVE, Horst, *et al.* [2017], Metric geometries in an axiomatic perspective, in: *From Riemann to Differential Geometry and Relativity*, edited by L. Ji, A. Papadopoulos, & S. Yamada, Cham: Springer International Publishing, 413–455, doi: 10.1007/978-3-319-60039-0_14.
- PASCH, Moritz [1882], *Vorlesungen über neuere Geometrie*, Leipzig: B. G. Teubner.
- PEANO, Giuseppe [1889], *I Principii di geometria logicamente esposti*, Turin: Fratelli Bocca.
- [1894], Sui fondamenti della Geometria, *Rivista di Matematica*, 4, 51–90.
- PESTALOZZI, Johann Heinrich, COOK, Ebenezer, *et al.* [1894], *How Gertrude Teaches Her Children. An attempt to help mothers to teach their own children and an account of the Method, a report to the Society of the Friends of Education*, Syracuse: C. W. Bardeen.
- PIERI, Mario [1896], Un sistema di postulati per la geometria proiettiva astratta degli iperspazî, *Rivista di Matematica*, 6, 9–16, reprinted in [Pieri 1980, 83–90].
- [1898], I principii della geometria di posizione composti in sistema logico-deduttivo, *Memorie della Reale Accademia delle Scienze di Torino (serie 2)*, 48, 1–62, reprinted in [Pieri 1980, 101–162].

- [1899], Review of [Ingrami 1899], *Rivista di Matematica*, 6, 178–182.
- [1900a], Della geometria elementare come sistema ipotetico deduttivo: Monografia del punto e del moto, *Memorie della Reale Accademia delle Scienze di Torino (serie 2)*, 49, 173–222, reprinted in [Pieri 1980, 183–234], translated into English in § 8 of [Marchisotto, Rodríguez-Consuegra *et al.* 2020].
- [1901], Sur la géométrie envisagée comme un système purement logique, in: *International Congress of Philosophy 1900-1903*, Librarie Armand Colin, vol. 3, 367–404, doi: 10.5840/wcp11901313, delivered by L. Couturat, reprinted in [Pieri 1980, 235–272], translated into English in [Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 4].
- [1906], Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive: Discorso per l'inaugurazione dell'anno accademico 1906-1907 nella Reale Università di Catania, *Annuario della Reale Università di Catania 1906-1907*, 23–82, reprinted in [Pieri 1980, 389–448].
- [1908], La geometria elementare istituita sulle nozioni di “punto” e “sfera”, *Memorie della Reale Accademia delle Scienze di Torino (serie 3)*, 15, 345–450, reprinted in [Pieri 1980, 455–560], translated into English in [Marchisotto & Smith 2007, § 3].
- [1980], *Opere sui fondamenti della matematica*, Rome: Cremonese.
- POINCARÉ, Henri [1899], La logique et l'intuition dans la science mathématique et dans l'enseignement, *L'Enseignement mathématique*, 1, 157–162.
- ROBERTS, David Lindsay [2014], History of tools and technologies in mathematics education, in: *Handbook on the History of Mathematics Education*, edited by A. Karp & G. Schubring, New York: Springer, 565–578, doi: 10.1007/978-1-4614-9155-2_28.
- ROWE, David E. [1989], Klein, Hilbert, and the Göttingen Mathematical Tradition, *Osiris*, 5(1), 186–213, doi: 10.1086/368687.
- [2018], *A Richer Picture of Mathematics: the Göttingen tradition and beyond*, Cham: Springer, doi: 10.1007/978-3-319-67819-1.
- RUSSELL, Bertrand [1903], *The Principles of Mathematics*, Cambridge: Cambridge University Press.
- SÉGUIN, Édouard [1843], *Hygiène et éducation des idiots*, Paris: Jean-Baptiste Baillière, citation from the reprint in *Premiers mémoires de Séguin sur l'idiotie (1838-1843)*, Paris: Aux bureaux du Progrès médical & Félix Alcan, 1897.

- SMITH, David Eugene [1919], Introductory course in mathematics, *The Mathematics Teacher MT*, 11(3), 105–114, doi: 10.2307/27949665.
- SMITH, James T. [2010], Definitions and nondefinability in geometry, *The American Mathematical Monthly*, 117(6), 475–489, doi: 10.4169/000298910X492781.
- SPENCER, William George [1860], *Inventional Geometry; A series of problems intended to familiarize the pupil with geometric conceptions and to exercise his inventive faculty*, New York: Appelton & Co.
- VON STAUDT, Georg Karl Christian [1847], *Geometrie der Lage*, Nuremberg: Verlag der Fr. Korn'schen Buchhandlung, edited and translated into Italian in [von Staudt 1889].
- VON STAUDT, Karl Georg Christian [1889], *Geometria di posizione*, Turin: Fratelli Bocca, edited and translated by M. Pieri, 1889.
- YOUNG, John Wesley [1911], *Lectures on Fundamental Concepts of Algebra and Geometry*, New York: The Macmillan Company.