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Thales, the "Pythagorean Theorem", and Technological Context

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In order to take a fresh look at Thales geometrical insights, I will place him in the broader, technological context of 6th century BCE Ionia. I start with Aristotle's report about Thales and the early philosophers who he claims were—to use my terms—*source* and *substance* monists. I argue that this view is testimony of their *modular* thinking. I defend the reliability of Aristotle's report by a new set of arguments[1] pointing out that examples of modular thinking were all around Thales, Anaximander, Anaximenes, Heraclitus, Pythagoras/Pythagoreans and interpenetrated their thoughts—in monumental stone temple architecture, in the production of coins in the monetization of their society, in the use of the gnomon, and even connected with the production of industrial textiles. Could it be that Thales *inferred* an underlying module of all things—he called it "water" (ὕδωρ)—by analogy with these other examples surrounding him? Everything that appears, according to Aristotle, are only alterations or modifications of this underlying nature or unity, because this unity always persists—this is the principle of modular thinking.

Now, had Thales held such a view, it seems difficult to avoid the question "*How* does this happen—*how* does 'water' flow shapelessly in a cup at one moment, then invisible as air, as fire at the stove, and yet sometimes hard as marble?" And what I propose to explore is the possibility that Thales' forays in geometry sought to identify the underlying *structure* of "water" out of which all other appearances were built, re-packaged and re-combined, and that Thales' plausibly reached the conclusion it was the

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[1] These case studies are examined in greater detail in my essay, Hahn (forthcoming).

right triangle. Accordingly, I will invite us to imagine the diagrams[2] corresponding to the reports that Thales in Egypt measured the height of a pyramid, measured the distance of a ship at sea, and place them alongside the diagrams that display the geometrical propositions with which Thales is also credited. Having done this, the reader can see that all, or mostly all, deal not only with right triangles but with *similar right-angled triangles*. And since one line of proof, preserved by Euclid (VI, 31), demonstrates the so-called Pythagorean theorem by similar right triangles, I argue it is plausible that Thales visualized, in a less sophisticated form, this line of reasoning for the famous theorem *because* it is a consequence of similar right triangles. And the "Pythagorean theorem" by this line of reasoning shows that the right triangle is the fundamental geometrical figure. So, the case I explore is what I have been calling the "Lost Narrative," the one that connects the reports about Thales' geometrical insights with his speculations about the underlying unity of nature.

I- Aristotle's Account of the Origins of Greek Philosophy is Historically Appropriate

Aristotle claims in the *Metaphysics*, 983b7 f, that most of the earliest philosophers posited a single underlying material principle of which all things consist, from which they first come (ἐξ οὗ γίγνεται) and into which on their destruction they are ultimately resolved (εἰς ὃ φθείρεται τελευταῖον); the essential nature persists although modified by its affections (τῆς μὲν οὐσίας ὑπομενούσης τοῖς δὲ πάθεσι μεταβαλλούσης). Thus, there is no real change since nothing else is generated or destroyed (οὔτε γίγνεσθαι οὐθὲν οἴονται οὔτε ἀπόλλυσθαι), since this primary nature always exists (ὡς τῆς τοιαύτης φύσεως ἀεὶ σωζομένης). These speculations about nature I will call "metaphysical speculations." The idea that there is some one entity from which all appearances arise is what I mean by the phrase "source monism"; and the idea that all appearances are only alterations or modifications of the underlying nature, since that nature always persists (ἀεὶ γὰρ εἶναί τινα φύσιν ἢ μίαν [...] ἐξ ὧν γίγνεται τἄλλα σωζομένης ἐκείνης) is what I mean by the phrase "substance monism." Let me be clear that "substance monism" is an example of modular thinking. Modular thinking introduces some basic unity that alters without changing; all things related to the module appear as modifications of that underlying unity since the underlying unity always persists.

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[2] To any who doubt that Thales was making diagrams, I commend to you my discussion of this problem in Hahn 2017a, p. 32-41, where Greek archaeological evidence contemporaneous and pre-dating the 6th century is evaluated.

Since the influential work of Cherniss,[3] Aristotle's account of the earliest philosophers has been cast into doubt. It is certainly true that at times Aristotle is an unreliable doxographer, but is he mistaken about *this* point? Most recently, Graham[4] has argued that he is because there is no evidence for "substance monism" either before or contemporaneous with Thales—that is the 6th century BCE—and thus attributing such a view to the earliest philosophers is "historically inappropriate."[5] The trouble with Graham's claim is that he investigated too myopically; if we now re-view the Ionian thinkers in their technological context, we will see they were surrounded by exemplars of modular thinking, a reflection on some of the surviving fragments reveals that these modular technologies interpenetrated their thoughts, that Aristotle's report is certainly *not* historically inappropriate, and quite likely spot on. Seen from this point of view, Thales' positing of such an underlying unity might very well have been an *inference* in metaphysical speculations about the underlying unity in nature, an innovation like that of the architects, coin-makers, gnomon-users, and wool felters.

The great limestone temple to Hera at Samos (Dipteros I) began sometime between 580-570 BCE, the limestone stage of the dipteral temple of Apollo at Didyma began shortly after 570, and the marble dipteral temple to Artemis at Ephesus, one of the seven wonders of the ancient world, began around 560.[6] Both Thales and Anaximander lived through the middle of the 6th century just when stone temple architecture began in their very backyard. Vitruvius reports that the archaic Ionic architects needed a module, in terms of which all the other architectural elements were reckoned as multiples or submultiples. The **module** was **column-diameter**.[7] Thus, in order to insure that the aesthetic effects would appear as planned when the building was scaled-up to monumental size, a basic unit—column-diameter—was identified,

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[3] Cherniss 1935.

[4] Graham 2006, p. 52-66. He uses Cherniss' phrase "material monism" to express what I have called both "source and substance monism." Because neither Cherniss nor Graham objected to the idea of "source monism," by rejecting MM they throw out the baby with the bathwater; they do not wish to challenge the appropriateness of "source monism" given the facts that both the Egyptian cosmologies identify the eldest god as Nun, the primordial ocean from which all things arise, and the Babylonian cosmologies begin with Apsu and Tiamat, the sweet and bitter oceans from which all things come. Their concerns are accepting what I have termed "substance monism," that there could be an underlying nature that always persists despite the diversity of appearances.

[5] Cf. Graham 2006, "The Case against Material Monism," p. 52-65.

[6] Cf. Hahn 2001, chapter 2.

[7] Vitruvius, III, 3, 7.

and then column-height was calculated, for example, at 9x the column-diameter, and the other elements such as the length and width of the stylobate, the height of the architrave and entablature, and so on, were calculated according to rules of proportion ultimately in column-diameter modules.[8] Anaximander imagined the size and shape of the earth by analogy with a 3x1 column-drum, a stone three times as wide as it was deep.[9] When he announced the distances to sun, moon, and stars that he imagined as fiery wheels encased in compressed (felted), evaporated moist-air that concealed the wheels,[10] he did so in terms of earth-diameters: 9/10 to the stars, 18/19 to the moon, and 27/28 to the sun, each wheel being 1 module thick. Anaximander was not only making use of a modular technique but moreover explicitly used the architect's module: earth = 1 column-diameter.[11]

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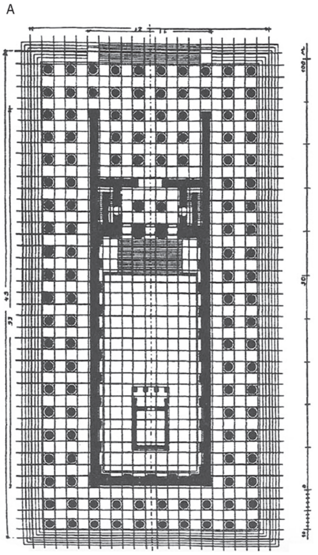
[8] Cf. Hahn 2003, p. 106-110, for the debate between architect-excavators Wesenberg and Krischen as to whether the proportional rule is 9x or 10x (*i.e.*, 9+1) the lower column diameter. It all depends upon where, precisely, on the lower diameter the module is identified. The columns taper upward as an optical corrector; in order for the column to look straight and uniform from the distance, it cannot actually be made straight and uniform. But where precisely on the column is it? Cf. also chapter 2 in Hahn 2010.

[9] Hippolytus, *Refutatio Omnium Haeresium*, I, 6, 3; and Ps. Plutarch, *Stromata*, 2. But, since the columns display *entasis*, that is, taper towards the top of the column shaft, the questions that remains is *where* precisely on the column was the module identified. While archaeologist-excavators debate precisely "where," the general consensus is that it is on the "lower column-diameter"—cf. Hahn 2010, p. 48-51.

[10] Hippolytus, *Refutatio Omnium Haeresium*, I, 6, 4-5.

[11] Cf. Hahn 2001, p. 181-191 for debates over the cosmic numbers, and also Hahn 2010, p. 60-64.



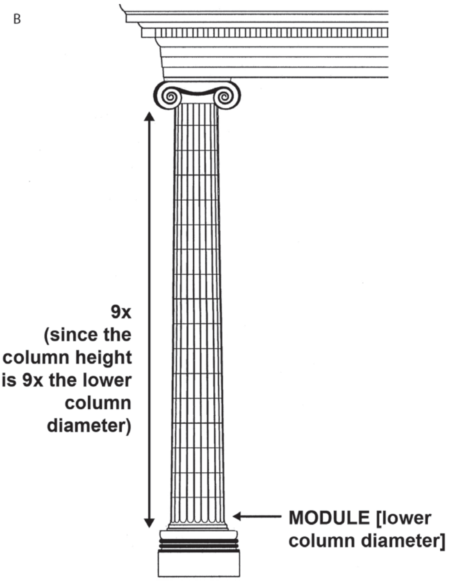


Figure 1A: Temple Architecture and Anaximander's Cosmos. Temple of Apollo at Didyma Grid Plan.
Figure 1B: Temple Architecture and Anaximander's Cosmos. The modular formula.

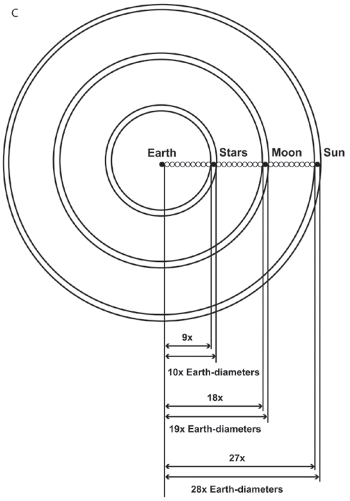


Figure 1C: Temple Architecture and Anaximander's Cosmos. Anaximander's Cosmos Plan-View.

The column, according to Homer, Hesiod, and Pindar had symbolic meaning either separating or connecting earth and heaven.[12] The modular formula in the archaic period for great intervals of time and great distances were reckoned in terms of the number "9".[13] Homer's *Iliad* opens in the 9th year of the war to be ended in the 10th: 9+1.[14] Hesiod measures the cosmos by asking us to imagine the dropping of an anvil from the highest heaven; it would fall 9 days and nights arriving on the 10th (9+1), and the distance from the gates of Hades to the depth of Tartarus is also supplied as falling 9+1 anvil days.[15] Odysseus wanders for 9 years following the fall of Troy finally returning in the 20th year: 9+1 + 9+1 (repeated). For the Ionic architects, heaven and earth are separated by 9x the lower column diameter; for Anaximander the distances to stars, moon, and sun are expressed by the same formula—9+1 earthly/column-diameters to the stars, +9 to the moon, +9 to the sun. It is not simply that Anaximander was living among the architects who were using modular techniques to make their *thaumata*; Anaximander imagined the cosmos in modular terms. Perhaps he used the architect's module because he came to imagine the cosmos as cosmic architecture. At all events, the key point is that the success of monumental architecture required a module so that the planned aesthetic effects—comparative height of columns to length and width of the stylobate, the architrave to the entablature, and so on—would be produced when the smaller model or diagram was scaled-up to gigantic size, that is, enlarged proportionately.[16] And here we have microscopic-macroscopic reasoning: the big world is imagined as the small world *scaled-up proportionally*, because the big world and small world share the *same structure*.

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[12] The column separates heaven and earth in Homer (*Odyssey*, I, 53) and Hesiod (*Theogony*, 517-519), but joins them in Pindar (*Pythian Odes*, I, 39-40).

[13] Cf. Hahn 2001, p. 173-174.

[14] Homer, *Iliad*, II, 327-329, and I, 52-54; *Odyssey*, XIV, 240-242.

[15] Hesiod, *Theogony*, 723-726.

[16] No models architecturally useful have survived, nor have any diagrams, perhaps made on a large surface such as an animal skin. Consequently, there is some doubt about whether models were made though it is my opinion that there were models as part of the design process—given the Egyptian evidence from whom the archaic Ionic Greeks learned much about monumental stone building (cf. Hahn 2001, chapter 3), and the idea that a tyrant or aristocratic patron would agree to finance arduous work decades-long without seeing and approving a model of the finished product strikes me as unlikely.

Both Thales and Anaximander used a gnomon to make measurements, the former to measure the height of a pyramid[17] and the latter to make a seasonal sundial.[18] Our testimonies claim that Thales placed a gnomon—a vertical rod—at the end of the pyramid's shadow, and reasoned that as the length of the shadow was to the height of the gnomon, so the length of the pyramid's shadow was to the pyramid's height. Anaximander is credited with making a *seasonal* sundial and setting it up in Sparta; by its varying shadow-lengths through the course of the year, Anaximander identified the summer and winter solstices, and even the equinoxes. In both cases, the **gnomon** serves as a **module**, its shadows altering without the gnomon itself changing.

Thales is credited with measuring the pyramid height when the shadow was equal to its height (**fig. 2A**) and also when it was unequal but proportional (**fig. 2B**). Note, in both cases, the technique requires that the module remains the same, only its shadows appear differently,[19] and it requires imagining in terms of similar right-triangles.

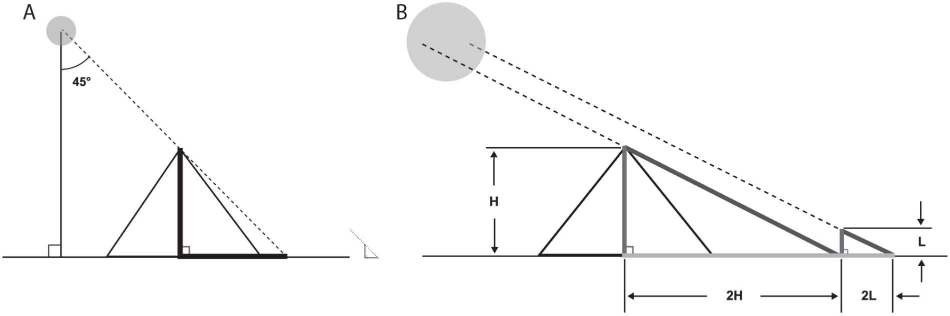

Figure 2: A. Measuring the pyramid height when shadow equals height. **B.** Shadow unequal but proportionate.

Anaximander set up a gnomon in Sparta. How did it look? We do not know, but the diagram that has to burgeon in his mind had to have certain features. I speculatively place his gnomon on a column drum, but surely it could have been set up in an open space.[20] Since he imagined the shape and size of the earth by analogy with a column-drum, the shadows on the drum-face would be analogous to the shadows on the earth itself, and perhaps this technique was influential to how Anaximander made the first

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[17] Plutarch, *Moralia*, II, 1, 353.

[18] Cf. DK 12A1 on the authority of Diogenes Laertius, and also in the *Suda*, *s. v*.

[19] Cf. the detailed exegesis of these two different shadow measurements in Hahn 2017a, p. 97-107 when shadow equals pyramid height, and p. 107-116.

[20] For the long discussion of the seasonal sundial, cf. Hahn 2010, p. 145-167.

map of the *oikumenē*, a *pinax* of the inhabited earth. I set the gnomon up in the middle of a circular drum-face where the *empolion* would be. Let us call the shortest shadow each day "local noon" and mark it with a dot. Plotting the shortest shadows each day throughout the year will create a straight-line north-south (**fig. 3A**). Now, (**fig. 3B**), the shortest of all short shadows during the year identifies the summer solstice; the longest of all short shadow markers identifies the winter solstice.

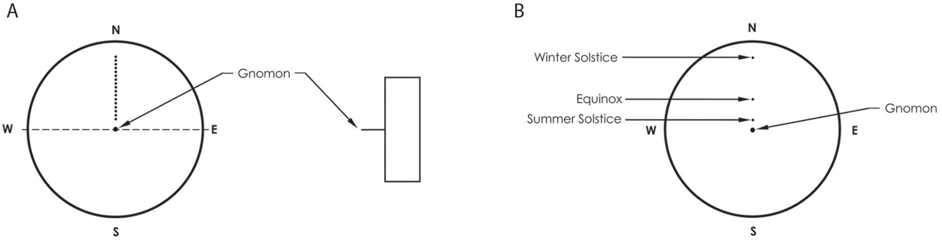

Figure 3: A. Anaximander's sundial, identifying local noon shadows. **B.** Isolating local noon shadows for solstices and equinox.

It just so happens that the equinox is not identified midway between these markers, but if one sighted the sunrise or sunset on summer and winter solstices from the sundial's west side (in morning) or east side (in the evening) and then bisected the angle created by 'A' the sunrise on summer solstice, and 'B' the sunrise on winter solstice, Anaximander could have identified the equinox(s) more precisely.[21] In any case, the east-west line of the equinox is perpendicular to the north-south line of the local noon shadows.

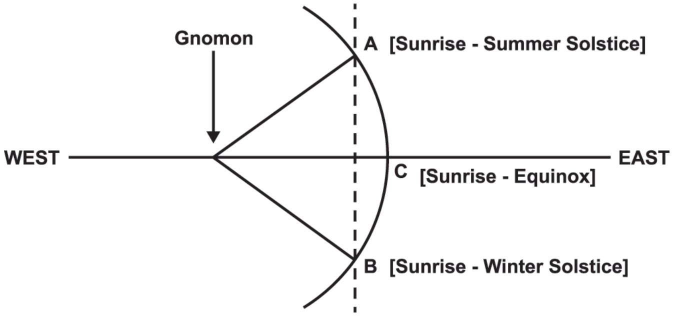

Figure 4: Bisecting the angle from the gnomon to the summer and winter solstices sunrise identifies the equinox.

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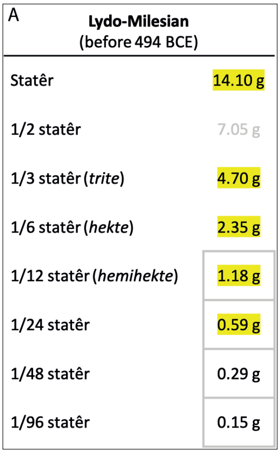
[21] It is interesting to consider that, since Anaximander is credited with making the first Greek map of the inhabited earth, and imagined the earth as a flat disk with ocean running round its circumference, the shadows on the circular sundial face would be microcosmically similar to the proportions on the map itself.

The invention of coinage introduces the counter-intuitive idea of a single substance underlying the plurality of things manifest to the senses, and it too, is an exemplar of modular thinking.[22] The Lydian invention found its way into Ionia in general and Miletus in particular by the end of the 7th century and yielded thousands and thousands of coins—electrum, gold, and silver.[23] The **module** was the ***statêr***, and it weighed approximately 14 grams and all the other denominations were reckoned as sub-multiples of that module. Within a monetized society, money becomes the source and substance of everything. Since money, as a standard of measure, provides an exchange value for all things, it can, in a metaphorical and analogical sense, become all things. Heraclitus expressed this idea precisely when he proclaimed that "All things are an exchange for fire, and fire for all things, just as gold for goods and goods for gold."[24] Keeping in mind that Aristotle identifies Heraclitus along with Thales and Anaximenes as a "substance monist"—though Heraclitus claimed it was fire (πῦρ), not water or air—here the underlying unity is identified by the analogy with the module of coinage. Substance monism (*i.e.*, fire) is illuminated by the modularity in coinage.


A

Lydo-Milesian (before 494 BCE)	
Statêr	14.10 g
1/2 statêr	7.05 g
1/3 statêr (*trite*)	4.70 g
1/6 statêr (*hekte*)	2.35 g
1/12 statêr (*hemihekte*)	1.18 g
1/24 statêr	0.59 g
1/48 statêr	0.29 g
1/96 statêr	0.15 g

Figure 5A: Lydian-Milesian Standard.

Figure 5B: The Statêr as Module.

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[22] Here I am indebted to Seaford 2004.

[23] Cf. the recent study by Hilbert 2018. The Milesian production of electrum coins from *ca* 600-530 BCE numbers in the tens of thousands of kilos, with the smallest coins weighing approximately 0,16 grams. The silver production, most heavily represented in the period of 560-546, roughly the time of Kroises' preparations and war with the Persians contributed also an enormous amount of coins in circulation in the last period of the lives of Thales and Anaximander.

[24] DK 22B90, preserved by Plutarch, *De E apud Delphos*, 8, 388D.

Aristotle (*Politics*, 1259a f.) shares with us the legend that Thales made a meteorological prediction of a bumper crop of olives for the coming season, and bought out all the local olive presses in anticipation. When the huge crop resulted, Thales rented out the presses for a neat profit. Had Thales done so, he would almost certainly have been paid a large share of Milesian *statêrs*. Thus, again, for both Heraclitus and Thales, all around the world of the Ionian Greeks, coinage provided an unchanging module in terms of which all other things are estimated as multiples or submultiples.

Finally, we have a report on Anaximenes from the trustworthy Theophrastus,[25] that -moist-air (ἀήρ) was the underlying substrate—the module—and all other appearances were rarefactions or condensations of it. All other appearances were alterations of it. Anaximenes appeals to the analogy of the felting of wool (πίλησις) to make this idea clear.[26] The process involves the shearing of sheep to get the wool, and if a design is desired, colored wool is fashioned into appropriate shapes and placed on a reed mat—the process requires an underlying substratum, that is, the mat. On top of the colored wool, the white wool is piled up and then sprinkled with water, and then rolled up—compressed—in the mat, and then compressed further by rolling the mat back and forth across the floor with the pressure of bodily weight upon it.

The doxographical reports from Simplicius, Hippolytus, Aetius, and Ps. Plutarch are in agreement that when air is rarefied it becomes fire, and when it is condensed, it becomes wind, clouds, rain, river and streams, earth, and finally stone. The process is illuminated by Anaximenes by appeal to the felting of wool. Cosmic diversity is accounted for by different levels of compression of the modular substrate. Here, the substance monism of air is clarified by the process of producing industrial textiles. What distinguishes appearances is the degree of compression on the module.

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[25] The point of contention is not whether Theophrastus is *always* reliable, any more than whether Aristotle is always reliable. Clearly, Aristotle is not, but my point is that in *this* case Aristotle's reporting seems to be spot on; Theophrastus is our principal source for the Presocratics, and at least in *this* case, Graham's proposal for GST is undermined by Theophrastus. Graham claims that before Parmenides a claim for "material monism" (I emphasize the issue is "substance monism") is impossible. As Zhmud pointed out in his review of Graham, Graham is mistaken when he proclaims that there is no fragment that explicitly supports Aristotle's report about material monism, and at least in Anaximenes' case we have a fragment of Theophrastus that unambiguously ascribes to air a role of underlying substratum that undergoes a series of qualitative transformations (fr. 2 Diels = fr. 226a FHSG). Graham quotes this fragment as Simplicius, *Physics*, 24, 26-28—A5 without saying that it comes from Theophrastus.

[26] Cf. Hahn 2017b for full photo details.

Figure 6A: Felting wool by the ancient method. Cloud-like wool piled up.

Figure 6B: Felting wool by the ancient method. Moistened by sprinkling water/rain.

Figure 6C: Felting wool by the ancient method. Compressed.

We can see in these four cases that modular thinking surrounded the Ionian communities in general and the Ionian thinkers explicitly: modular thinking is interwoven in the surviving doxographical reports; it proposes an underlying unity that alters without changing. Aristotle's claim, then, that the early philosophers were source and substance monists is historically appropriate; Graham is mistaken on this crucial point. From where did Aristotle learn about Thales? Cherniss thought only by oral transmission, but it might very well be, as Snell[27] argued, Claussen[28] echoed, and Panchenko[29] most eloquently reasoned that Hippias of Elis was the likely source.[30] When we review Aristotle's report against the technological context of 6th century Ionia—the modular thinking in architecture, gnomon, coinage, and even the felting of wool—perhaps we can see the plausibility of how Thales *inferred* an underlying module in nature—water—on analogy with these other examples of modular thinking. I now turn to explore the reports about Thales' explorations in geometry. Could it be that his inquiries in geometry fit into this broader picture? Could it be that they were part of the search for the *structure* of the module—water? This would mean he was searching for the fundamental geometrical figure (= module) out of which all appearances were constructed by re-packaging and re-combination of that figure. Can that case be made, even if only circumstantially? The substance monist maintains that there is a single underlying nature or unity that alters without changing. Could the geometrical building block of water—the right triangle—have been part of his modular thinking?

II- THALES' GEOMETRY AND SO-CALLED "PYTHAGOREAN THEOREM"

Plato makes Timaeus present a view of earlier theories of the underlying principles of the cosmos that is consistent with Aristotle's report at *Metaphysics*, 983B f. At *Timaeus*, 48E, a fresh start is made in the account of the underlying principles of the cosmos and its creation, where Timaeus will add to Form and Copy (and Demiurgos),

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[27] Snell 1944, p. 170-182.

[28] Classen 1965, p. 175-178.

[29] Panchenko 2005, p. 77 f.

[30] I urge the readers to carefully review the elegant arguments by Panchenko that Hippias, or even Oenopides or Hippocrates of Chios is/are Aristotle's likely source. But, of course, I am still stuck with the problem of explaining why there are no explicit statements from these sources that Thales held this doctrine or project. Still, when Proclus mentions, on the authority of Eudemus, that Thales was responsible for many insights into geometrical propositions, some of which he investigated empirically and some more generally, so much also has been omitted.

the Receptacle or nurse of all Becoming. As he proceeds, he segues to the elements and points to a difficulty in accounts for those that supposed the underlying unity is "water" (Thales?), "air" (Anaximenes?), or "fire" (Heraclitus and Hippasus?), although no individual is mentioned by name. Timaeus raises the objection that trying to identify the underlying unity with any of the elements is self-stultifying since at one moment we see water or liquid things, and at another fiery things, or invisible airy things—nothing endures and so, to be precise, we cannot even point to any "that" but must instead refer only to "suchlike." Thus, his argument suggests that those who held that water, air, or fire was the underlying nature or unity were advocates of what I have been calling "substance monism," that despite the diversity of appearances, there was one and only one underlying nature or unity that did not perish. Then, Timaeus proceeds to arguments for constructing the cosmos out of a fundamental geometrical figure—the right triangle. And so, the metaphysical topic of accounting for an underlying principle, nature, or unity of the cosmos flows directly to the geometrical task of constructing the cosmos out of the fundamental geometrical figure. Could this interconnected project have been the insight of the Pythagoreans? Timaeus comes from southern Italy where the Pythagoreans transmitted their ideas, and they are also credited—unlike Empedocles and Alcmaeon—with geometrical activities including the sum and substance of Euclid's Book II (as well as VII, VIII, IX). As I have argued elsewhere,[31] along with Zhmud,[32] Pythagoras himself may well have played a leading role in geometrical investigations. But where and how did Pythagoras get the impulse to turn there? It is not unreasonable that Thales, Anaximander (who is credited with producing an Outline of Geometry),[33] and members of their Milesian school stimulated such interests.

Now I propose to reverse engineer my argument. Plato's *Timaeus* 53C f. memorializes the project of building the cosmos out of right triangles. Where did it come from? Might it have come from Pythagoras/Pythagoreans, who got the project from Thales?

> In the first place, then, it is of course obvious to anyone that fire, earth, water, and air are bodies; and all bodies have volume (βάθος). Volume, moreover, must be bounded by surface (ἐπίπεδον), and every surface that is rectilinear (ὀρθή) is composed of triangles. Now all triangles derive their origin (ἄρχεται) from two triangles, each having one right angle and the other acute (*i.e.*, isosceles and scalene).

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[31] Cf. Hahn 2017a, p. 136-137.

[32] Zhmud 2012, p. 256-257.

[33] Cf. *Suda*, *s.v.*, ὅλως γεωμετρίας ὑποτύπωσιν ἔδειξεν.

All the objects in the cosmos are made out of four basic elements fire, earth, water, and air, and every object occupies a space bounded by its surfaces. And every surface (= every flat space or figure) dissects into triangles, and inside every triangle there are two right triangles, isosceles and scalene. And so every body—the whole cosmos—is constructed out of right triangles. The right triangle is the fundamental geometrical figure. Here is the argument in diagrams:

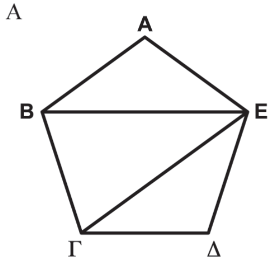

Figure 7A: The argument of Plato's Timaeus 53C in diagrams. All figures dissect into triangles.

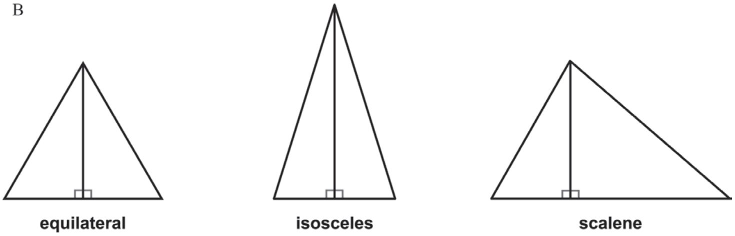

Figure 7B: The argument of Plato's Timaeus 53C in diagrams. Inside every triangle are two right triangles.

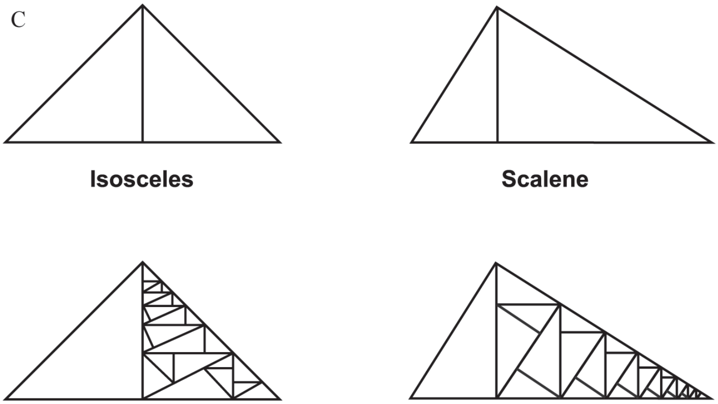

Figure 7C: The argument of Plato's Timaeus 53C in diagrams. All right triangles are either isosceles or scalene, and they divide from the right angle into similar right triangles ad infinitum.

Now once we realize that every triangle contains within it right triangles, we can build the cosmos out of equilateral triangles and squares, since each divides into right triangles. Below, are the elements, **fig. 8**, later identified with geometrical figures that came to be known as the “Regular Solids” or “Platonic Solids,” each built out of right triangles, set out on a flat surface and folded-up to make a volume. To “put together” (σύστασις) the regular solids—an achievement specifically credited to Pythagoras himself by Proclus[34]—equilateral triangles, and squares, are set out, and then folded up into volumes: this works only for 4, 8, and 20 equilateral triangles, and 6 squares.[35]

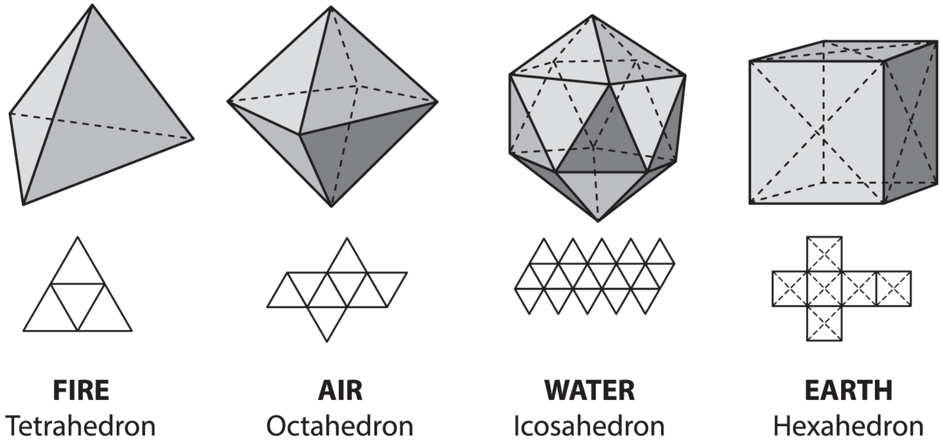

Figure 8: The elements are “molecules” for all bodies in the cosmos, built of scalene and isosceles right triangles.

Plato makes Timaeus of Locri, of southern Italy, narrate this account, and since he is talking to a group that includes Socrates, the dramatic date of the dialogue is sometime in the second half of the 5th century. Again, identifying Timaeus, from southern Italy, certainly suggests Pythagorean influence. Now, let us agree that no one proposes a project of building the cosmos out of right triangles without a proof, or line of reasoning, that shows the right triangle is the fundamental geometrical figure. Was there a proof that Timaeus and Socrates knew that showed this? Apparently, such a proof was known because no objection is raised when the idea is presented. Could it have been the hypotenuse theorem, that is, the “Pythagorean theorem”?

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[34] Proclus 65-66 (Morrow, 53).

[35] Here, I have left out the dodecaheadron, made of 3 regular pentagons, It is not identified with sny of the four elements.

Now, perhaps a concern could be raised why we do not first explore the evidence of cuneiform tablets from Babylon that shows both a geometrical diagram and so-called triples, hence a metric interpretation as well, of the Pythagorean relation, to unfold the Greek discovery, and perhaps their indebtedness to the Babylonians. I have already argued at length why this is not a promising path.[36] It is certainly true that we have evidence that pre-dates the time of Thales and Pythagoras by a millennium that shows the Babylonians had an understanding of it. But this expressed concern obfuscates a crucial point: *which* interpretation of the Pythagorean relation? Let me explain this crucial point a bit further. To proceed along this path will not help us understand the "Greek discovery" of the hypotenuse theorem. For suppose our investigation should begin by stating a familiar formula for the theorem: $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$ and we set off to try to understand it and how the Greeks might have understood it. This would be a non-starter for our project here because this formula is an algebraic equation and the Greeks did not have algebra! Moreover, we now know there are more than three hundred fifty different proofs of the Pythagorean theorem, each one revealing a different aspect of the right triangle.[37] So, the mistake in approach is to suppose that to know one line of thought to the Pythagorean proposition is to know other lines of thought leading to it. There is no evidence to suggest that the Babylonians investigated the right triangle as part of an inquiry into a metaphysical reality that underlies appearances. What the evidence does show for the Greeks is that they knew at least two proofs, because they are preserved by Euclid: Book I, 47, likely developed from Eudoxus, and the proof by similar triangles at VI, 31. Do either of them reveal, or could be interpreted to reveal, that the right triangle is the basic geometrical figure? Had Pythagoras or someone in his school produced a proof, it was almost certainly a less sophisticated form of either of these proofs, and the case to be explored now is that it sets out the lines of thought

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[36] Cf. Hahn 2017a, p. 9-12 I show the Cuneiform tablets in question and a discussion about them, also considering the issue of whether the Greeks—focused on Euclid Book II—presented geometrical explorations of Babylonian algebra, usually referred to as "geometrical algebra." I argue, contra van der Waerden, and with Unguru 1975, p. 171, that the Greeks were not involved with geometrical algebra, and the very idea is wrong-headed. As Unguru stated so eloquently: "Geometry is thinking about space and its properties. Geometrical thinking is embodied in diagrammatic representation accompanied by a rhetorical component, the proof. Algebraic thinking is characterized by operational symbolism, by the preoccupation with mathematical relations rather than with mathematical objects, by freedom from any ontological commitments, and by supreme abstractedness. 'Geometrical algebra' is not only a logical impossibility it is also a historical impossibility."

[37] Maor 2007.

in VI, 31 the proof of the hypotenuse theorem by similar right triangles. This proof is a consequence of understanding similar right triangles, and it can be interpreted to reveal that the right triangle is the fundamental geometrical figure. A survey of the oldest diagrams in Euclid, dating to medieval times, shows the most prevalent diagram (**fig. 9**).[38] Note, there are no figures at all drawn on any of the sides, only a perpendicular drawn from the right angle at "A" to the hypotenuse ΒΓ.[39] The perpendicular divides the triangle into two similar right-angled triangles, *ad infinitum*. And, we keep in mind that every surface can be divided into triangles, and inside every triangle are two right-angled triangles, and if we continue to divide from the right angle, we will divide, forever, every right triangle into two similar right triangles. *The right triangle is the module*; all other figures are constructed out of them because all figures reduce to them.

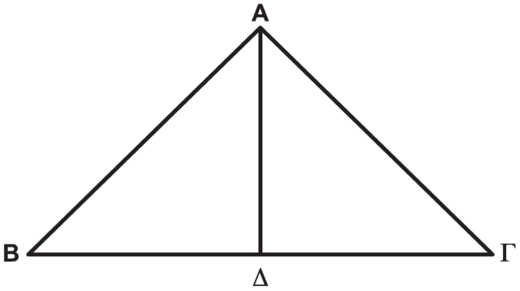

Figure 9: The most prevalent diagram of the Pythagorean Theorem at Euclid VI, 31 according to the medieval manuscripts.

Since Burkert's study,[40] a scholarly consensus has developed discrediting Pythagoras with the famous theorem on the grounds that the evidence is too late to be secure; the first explicit mention appears in Vitruvius' *Ten Book on Architecture*, dating to the 1st century BCE.[41] And yet, Vitruvius mentions that the Ionic architect of the archaic temple to Hera at Samos, Theodorus, and the archaic architects of the temple to Artemis at Ephesus, Chersiphron and Metagenes, wrote prose treatises more or less contemporaneous with the young Pythagoras, and no one has ever challenged

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[38] Cf. Hahn 2017a, p. 122; and Ken Saito's website http://www.greekmath.org. Cf. Reproduced Diagrams in Greek and Arabic Manuscripts, 2013, p. 94; and in Diagrams in Greek Mathematical Texts, 2011, p. 195.

[39] Ancient Greek diagrams conventionally have the right angle at "A" whereas in modern textbooks it is conventionally labelled "C".

[40] Burkert 1972.

[41] Vitruvius, IX, 6.

that claim made by Vitruvius.[42] Zhmud[43] has argued that Burkert was too hasty in his dismissal of Pythagoras the mathematician; we have evidence in the 5th, 4th, and 3rd centuries BCE connecting Pythagoras himself with mathematical activities, and even a diagram that might well have been connected with this theorem. While I think it more likely that Zhmud has this right, the arguments presented here follow the diagrams. Zhmud accepts that Pythagoras is likely connected with (i) the hypotenuse theorem, (ii) the Application of Areas theorem, and (iii) the "putting together" of the regular solids. Had Pythagoras done so, they form a picture of (i) identifying the fundamental geometrical figure, (ii) transforming all appearances from triangles into figures of different shapes and sizes, and so a cosmos of diverse appearances generated from a module (= the right triangle), and (iii) building the molecules (as it were) made of right triangles (= module), that are the building blocks of all other appearances. Now if we accept for this moment these reported achievements, placed together, they seem to be part of the project of building the cosmos out of right triangles; if so, how did this project burgeon in Pythagoras' mind? Of course, granting the exiguous nature of the evidence, it is impossible to say with any degree of certainty, but if we adopt the approach on which Plato takes his reader in the *Timaeus* (29D) we, too, can try to produce a "likely account." To do so, we shall investigate whether there were any ongoing projects and activities in geometry that may have directed Pythagoras into this line of thinking. Could Thales' forays in geometry have done so?

We have the report from Proclus on the authority of Eudemus, Aristotle's pupil who wrote the *History of Geometry*, that Thales went to Egypt and was the first to introduce geometry into Greece, crediting him with discoveries some of which he approached more empirically and some more generally.[44] Herodotus also reports that geometry was brought from Egypt to Greece through observation of land-surveyors.[45] What could Thales have learned from them? Herodotus tells us that the King divided the land into square plots, and each man was taxed on that area in the form of crops paid as tribute. Now any man who claimed that he was robbed of arable land by the Nile flood could petition the King, and the King would send out surveyors to re-measure the

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[42] Vitruvius, VII, 12.

[43] Zhmud 2012, p. 239-274.

[44] Proclus, 65, 7. Θαλῆς δὲ πρῶτον εἰς Αἴγυπτον ἐλθὼν μετήγαγεν εἰς τὴν Ἑλλάδα τὴν θεωρίαν ταύτην καὶ πολλὰ μὲν αὐτὸς εὗρεν, πολλῶν δὲ τὰς ἀρχὰς τοῖς μετ᾽ αὐτὸν ὑφηγήσατο, τοῖς μὲν καθολικώτερον ἐπιβάλλων, τοῖς δὲ αἰσθητικώτερον.

[45] Herodotus, II, 109.

land and diminish his tax-burden by the proportion that had been robbed. Is there any Egyptian evidence for the geometrical technique? There might well be.[46] In the *Rhind Mathematical Papyrus* problem #51 there is instruction to calculate a triangular plot of land. If one imagines, as the surveyor in the Nile valley would have occasion to do, that space is flat and every square plot can be divided into triangular areas, the pieces of land no longer arable could be reduced to triangular plots. The areas of those triangular plots could be reckoned by the formula in *RMP* #51, and the sum of those areas could be subtracted from the total area of the square. Seen in such a light, Thales was in a position to have learned, or confirmed, from the Egyptian surveyors, that the triangle was the fundamental geometrical figure, since all figures dissect into them. Perhaps this knowledge was the platform from which Thales began to think geometrically, and about triangles?[47]

In order to proceed in his explorations, Thales had to know that the angles of every triangle summed to two right angles. We learn from both Proclus, and Geminus, that the Pythagoreans proved it. But, from Geminus[48] we learn that the mathematicians before them—who could only be Thales and members of his generation—had theorized how there were two right angles in every species of triangle: equilateral, isosceles, and scalene. How did Thales do it? We are not told, but he might have made use of an approach that he was in a position to have learned in Egypt when he visited there, by means of RMP #51, a *triangle's rectangle*. Drop a perpendicular from a vertex to the opposite side of any triangle, and complete the two rectangles, **fig. 10**. We can see there are 4 right angles in every rectangle and each triangle-half must contain two right angles since each is half of the rectangle. If we subtract the two right angles at the base, we can see immediately that every triangle contains angles that sum to two right angles.

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[46] Cf. Hahn 2017a, p. 12-24 where the relevant geometrical problems of the *Rhind Mathematical Papyrus* are set out.

[47] There is also a possibility that Thales may have first learned or confirmed this geometrical insight while watching surveyors of the nearby Meander River. As Thonemann 2011, p. 95, has pointed out, as the Meander belched out soil as it made its way to the sea coast, there were debates about who owned this "new" land. Here, the surveyors were also, perhaps, in a position to divide the new land into triangles to discover their areas.

[48] Geminus is quoted in Apollonius' *Conica*; cf. also Heath 1921, vol. II, p. 318 f. ὥσπερ οὖν τῶν ἀρχαίων ἐπὶ ἑνὸς ἑκάστου εἴδους τριγώνου θεωρησάντων τὰς δύο ὀρθὰς πρότερον ἐν τῷ ἰσοπλεύρῳ καὶ πάλιν ἐν τῷ ἰσοσκελεῖ καὶ ὕστερον ἐν τῷ σκαληνῷ οἱ μεταγενέστεροι καθολικὸν θεώρημα ἀπέδειξαν τοιοῦτο: παντὸς τριγώνου αἱ ἐντὸς τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

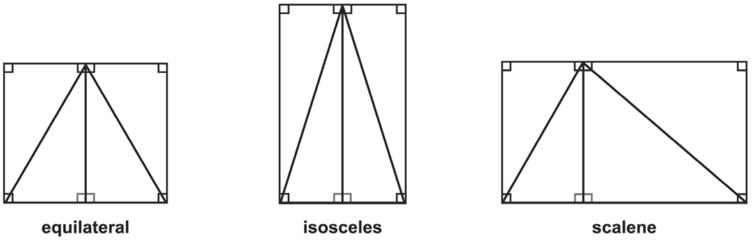

Figure 10: How Thales may have visualized that the angles of every triangle sum to two right angles.

Next, Thales is credited with measuring the height of a pyramid by its shadows, both when the shadow was equal to its height and when the shadow was un-equal but proportional (**fig. 2A** and **2B**). Both methods make use of a gnomon, and both require an understanding of similar right triangles. In the first case, when the gnomon's shadow is equal to its height, then this is the time of day when *every* vertical object casts a shadow equal to its own height. Thus, the pyramid's shadow is equal to its shadow length.49 In the second case, the shadow of the gnomon is to the gnomon's height as the shadow of the pyramid is to the pyramid's height, a case of proportional reasoning.

Thales is also credited with measuring the distance of a ship at sea. Had he made the measurement from the shoreline (**fig. 11A**), or even from a raised tower (**fig. 11B, C, D**), all required an understanding of similar right triangles.

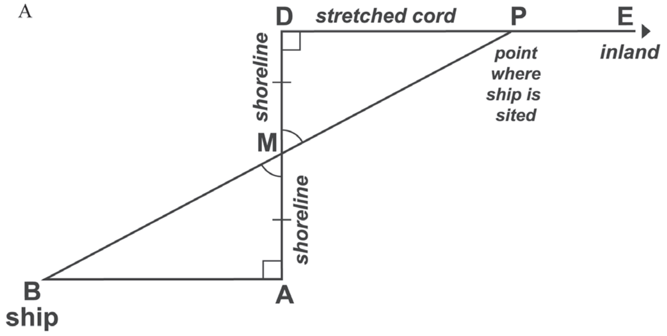

Figure 11A: Diagrams showing how Thales may have imagined measuring the distance of a ship at sea: from the shoreline.

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[49] This is an over-simplification and consequently may be misleading. Cf. Hahn 2017a, chapter 2. To be measured exactly, the shadow must be perpendicular to the pyramid's base, and the sun must be due east, west, or south, since the measurable shadow is mostly under the pyramid. And this happens only 4 days a year! If the shadow projection is not perpendicular to the side of the pyramid, then trigonometry would be needed to adjust for an exact measurement, and the Greeks at that time did not have trigonometry.

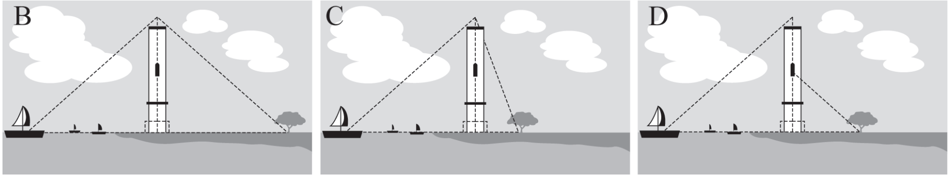

Figure 11B, C, D: Diagrams showing how Thales may have imagined measuring the distance of a ship at sea from a raised tower.

The Pythagorean theorem is a consequence of similar triangles. If one divides a right triangle from its right angle, two *similar right triangles* are produced inside of it, each of which has the same shape as the original triangle being divided, but are of different size. The isosceles right triangle divides into two equal similar right triangles (**fig. 12A**); the scalene divides into two un-equal similar right triangles (**fig. 12B**).

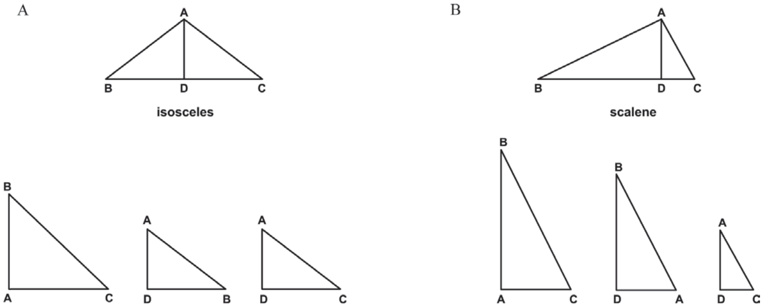

Figure 12: Dividing a right triangle from its right angle at A produces two similar right triangles ad infinitum. **A.** Isosceles, **B.** Scalene.

In addition, there are geometrical propositions with which Thales is credited, and the diagrams connected with them appear in **fig. 13**:

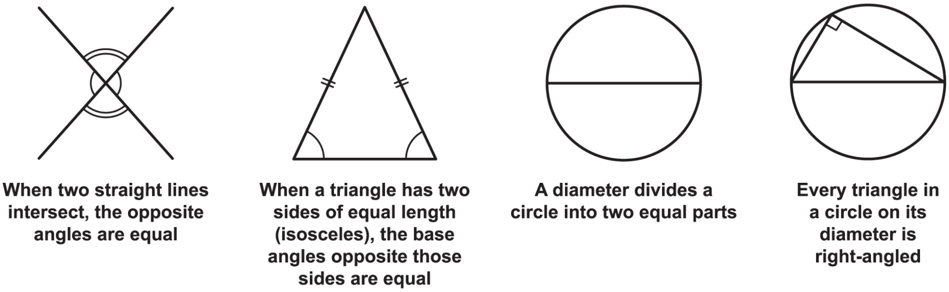

Figure 13: The diagrams of Thales propositions.

The mathematician Pamphila claims, according to Diogenes Laertius I, 24-25, that Thales was the first to inscribe a right-angled triangle in a circle, and sacrificed an ox upon it. What was so momentous about this achievement, we will consider at the end of this essay. Allman, Gow, and Heath agree that this must be what is later preserved by Euclid at III, 31, that every triangle inscribed in a circle *on its diameter* is right-angled. Here is, perhaps, how Thales may have done it. But note how the propositions "diameter bisects a circle," and "every isosceles triangle has the same angle opposite those equal sides," play important contributing roles. The line of thought must show that the angle at "A" is right, in the case of the isosceles right triangle (**fig. 14A**) and the scalene right triangle (**fig. 14B**).

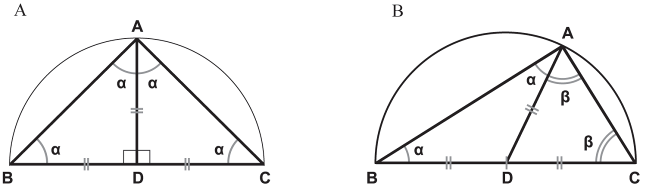

Figure 14: How Thales may have realized that every triangle inscribed in a circle on its diameter is right angled.

In both diagrams, BD, AD, DC are all radii of the circle BAC, and so must be equal in length. In **fig. 14A**, AD and AB must be of equal length, and since the angle ADB is right, and the two other angles must be equal since the sides opposite them are equal; angle "α" must be ½ right angle. The angle "β" must also be equal to ½ right angle for the same reasons, and thus "α"+"β" must equal 1 right angle. In **fig. 14B**, the situation is analogous. BD, AD, and DC are all of equal length since they, too, are all radii of circle BAC. There are three angles in the triangle ABC—angles "α", "β", and "αβ"—and angle "αβ" must be shown to equal 1 right angle. Now, since the angles of every triangle sum to two right angles: therefore, "α", "β", and "αβ" = 2 right angles, and consequently "αβ" must equal 1 right angle. Diogenes' report also continues by suggesting that some claim that this discovery was made by Pythagoras. But what seems more likely, in my estimation, is that Pythagoras followed the lead of Thales and used *the same diagram*, that is, the same lines of thought.

Now, had Thales been looking for the fundamental geometrical figure and come to the conclusion, following similar right triangles, that it was the right triangle, discovering that every triangle inscribed in a circle on its diameter was right would allow him to produce as many right-angled triangles as needed to further his investigations. If he now looked inside the right triangle to see if anything more was to be discovered, he would have immediately seen areal equivalence in the isosceles right triangle in a circle.

In Fig.15A, he could see that since BD, AD, DC are all equal in length, as radii of the circle ABC, the square on AD was equal to the rectangle made by BD, DC, namely that, it too, would be a square. In Fig.15B, in the scalene right triangle, the square on the perpendicular is equal to the rectangle made by length BD and width DC, but this would have to be confirmed empirically, by compass and straight-edge, analogous to how the masons worked at the temple site using caliper and straight-edge.[50]

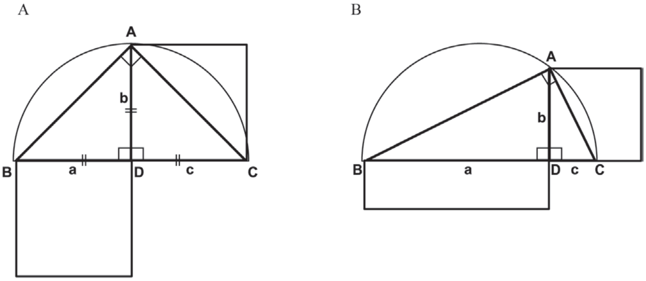

Figure 15: How Thales may have discovered the mean proportional. **A.** Isosceles right triangle, **B.** Scalene right triangle.

What is immediately obvious in the isosceles right triangle (**fig. 15A**) is that the perpendicular AD divides the hypotenuse into two equal lengths, and in a continuous proportion, or mean proportional (μέση ἀνάλογος): DC : AD :: AD : BC. In the scalene triangle, the relationship is the same, but now empirical measurement is needed to confirm that the rectangle made by the first and third lengths is equal in area to the square on the second. Could it be that this discovery suggested to Thales that when the basic figure continues to collapse, or expand, it does so in a pattern? And if there was such a pattern inside a right-triangle's collapse by division into similar right triangles, might there yet be other mean proportional relations in the right triangle? Had Thales looked, he would have discovered that there were two others.

The continuous proportion BC : AB :: AB : BD means geometrically that the square on AB (the second) is equal in area to the rectangle made by length BC (first) and width BD (third); and BC : AC :: AC : DC, thus the square on AC (the second) is equal to the rectangle made by length BC (first) and width DC (third). And here we can see in Figs.16A and B that the square on the hypotenuse (= the sum of two rectangles into which the hypotenuse was divided) is equal to the sum of squares on its other two sides. **The discovery or realization of these two mean proportional**

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[50] The Pythagoreans later proved it, now preserved in Euclid, II, 14 showing how to make a rectangle equal to a square which is a proof of the mean proportional.

relations among the sides of a right triangle is the visualization of one of the proofs the "Pythagorean theorem".

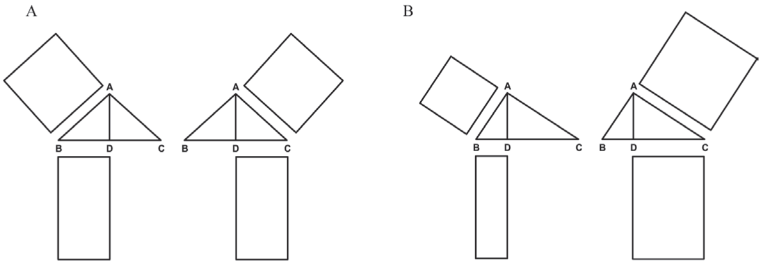

Figure 16: How the discovery of two mean proportionals is one of the visualizations of the Pythagorean Theorem for the isosceles right triangle in **A**, and the scalene right triangle in **B**.

Might Thales have visualized this? Had he been looking for the fundamental geometrical figure—the *structure* of water in reply to questions by his compatriots as to just how "water" appears so divergently, he may have reasoned that there must be some basic structure that by re-packaging and re-combining accounts for a cosmos of such different appearances given that there is only one substance that underlies all appearances because it never perishes. And had Thales realized it was the right triangle, might he have looked inside the right triangle to see what more could be discovered such as the patterns of collapse or expansion of the right triangle into the many objects that appear?

Imagine, again, the diagrams that almost certainly burgeoned in his mind in order to measure pyramid height by its shadows, or the distance of a ship at sea: they are all the same diagram! They all entail *similar right-angled triangles*, and reasoning by geometrical similarity, **fig. 17**:

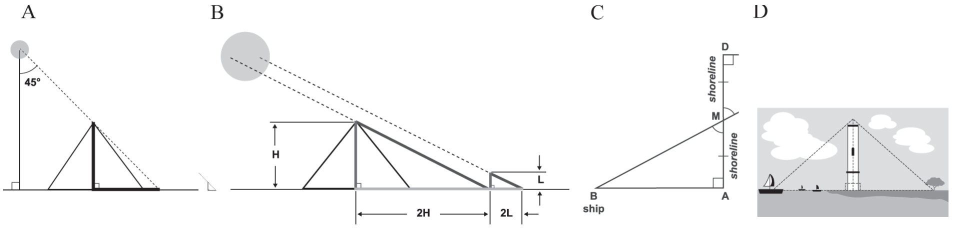

Figure 17: The diagrams associated with Thales' measurements of pyramid height (**A, B**) and distance of ship at sea (**C, D**) are all the same, only rotated in **C**.

Reviewed in this light, perhaps Thales visualized—not proved—the "Pythagorean theorem" as two mean proportionals, and Pythagoras took up the project bequeathed by Thales, and followed through the *same* diagram that every triangle inscribed in a circle on its diameter is right in order to produce his proof?

There is another confusion to be addressed. Some scholars have doubted that Thales made such a sacrifice on the grounds that this seems like confusing the legend, reported by Diogenes (VIII, 10-12), that Pythagoras made a splendid sacrifice of oxen upon his proof of the famous theorem. Moreover, some scholars have doubted the story about Pythagoras' sacrifice at all on the grounds that he was a vegetarian, and would hardly approve of the killing of animals. But let us take a fresh look from the lines of thought we have unfolded here. Had Thales visualized two mean proportionals through lines of thought of inscribing a right triangle in a circle, this geometrical knowledge would have been known *before* the middle of the 6th century, when young Pythagoras (born *ca* 570 BCE) was still a young man in Samos, not the documented older vegetarian in southern Italy. Had Pythagoras been aware of Thales' activities, perhaps it was *young* Pythagoras who produced the proof and made the sacrifice? Anyone who investigates the archaic temple to Hera at Samos (Dipteros I and II) and sees its gigantic altar some 60 meters in length realizes that the Samian way to thankfully acknowledge a divine inspiration would certainly have entertained a glorious sacrifice. And, having taken this approach to see the underlying project as a search for the fundamental geometrical figure, might Pythagoras's sacrifice be celebrating not simply his proof of the special character of the right triangle but, moreover, the proof of the fundamental building block of the cosmos?

And what of Thales' sacrifice that was thought to be a confusion of the report of Pythagoras's grand sacrifice? Suppose this "likely account" proposed here, that I have elsewhere called the "Lost Narrative,"[51] namely the lost connection between Thales' geometrical explorations and his metaphysical speculations about the underlying unity in nature that alters without changing—places Thales with his compatriots declaring that "water" is the underlying nature of all things. Surely some of his compatriots were incredulous and asked him how water could appear one moment like fire, then hard as stone, and so on. The possible answer we have explored is that all these appearances were formed out of combinations of right triangles, because all objects have surfaces, and every surface reduces to triangles, and inside every triangle are two right-angled triangles, and they will further divide from the right angle indefinitely into similar right triangles—thus, the right triangle is the module. But can we not anticipate the possible

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[51] Hahn 2017a, p. 2-3.

objection raised by some clever associate: "Well, this may be true of all rectilinear figures, but what of the circle? The round circle is not made out of pointy right triangles, is it?" And Thales answer was to invite the challenger to take another look at the diagram (**fig. 18**) showing every triangle in a circle on its diameter is right angled.

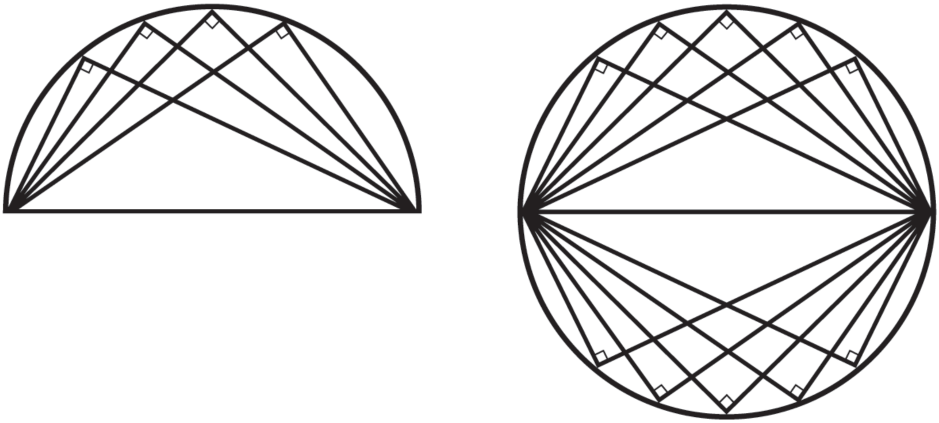

Figure 18: Plotting every possible right triangle on the diameter of a circle creates a geometric *loci*; even the circle is built out of right triangles!

Lo and behold, what it shows, as surprising as it must first seem to those unfamiliar with geometrical figures, is that the circle itself is in fact constructed out of right triangles! If one plots every possible right triangle in that circle, identifying each point where the circle is touched from each end of the diameter, one produces what modern mathematicians call a 'geometrical loci,' in this case, a circle. Could it be, then, that in showing that even the circle is made of right triangles, that indeed *all* figures and shapes were constructed out of right triangles, Thales had an exceptional reason to produce, in gratitude to the divine inspiration he clearly received, a splendid sacrifice?

Bibliography

Burkert W. (1972), *Lore and Science in Ancient Pythagoreanism*, Harvard [transl. E. L. Minar].

Cherniss H. (1935), *Aristotle's Criticism of Presocratic Philosophy*, Baltimore.

Classen J. (1965), "Bemerkungen zu zwei griechischen 'Philosophiehistorikern'", *Philologus*, 109, p. 175-181.

Graham D. (2006), *Explaining the Cosmos: The Ionian Origins of Scientific Philosophy*, Princeton.

Hahn R. (forthcoming), "The Metaphysics of Modular thinking, Substance Monism, and the Origins of Greek Philosophy: Architecture, Geometry, Coinage, and the Felting of Wool," in *Materia Philosophiae: The Material Dimensions of Greek Philosophy*.

Hahn R. (2020), "Architectural Technologies and the Origins of Philosophy", *ARCHAI*, 20, p. 1-29.

Hahn R. (2017a), *The Metaphysics of the Pythagorean Theorem: Thales, Pythagoras, Engineering, Diagrams, and the Construction of the Cosmos out of Right Triangles*, Albany.

Hahn R. (2017b), "Heraclitus, Milesian Monism, and the Felting of Wool," in E. Fantino, U. Muss, C. Schibert, K. Sier, (eds), *Heraklit im Kontext*, Berlin, p. 187-209.

Hahn R. (2010), *Archaeology and the Origins of Philosophy*, Albany.

Hahn R. (2003), *Anaximander in Context: New Studies on the Origins of Greek Philosophy*, Albany [co-authored with Dirk Couprie and Gerard Naddaf].

Hahn R. (2001), *Anaximander and the Architects: The Contributions of Egyptian and Greek Architectural Technologies to the Origins of Greek Philosophy*, Albany.

Heath T. (1921), *A History of Greek Mathematics*, 2 vols, Oxford.

Hilbert R. (2018), *Die Electronprägung von Milet*, Bonn (*Nomismata*, 9).

Maor E. (2007), *The Pythagorean Theorem: A 4000-Year History*, Princeton.

Morrow G. (1970), *Proclus: A Commentary on the First Book of Euclid's Elements*, Princeton.

Panchenko D. (2005), *Thales and the Origins of Theoretical Reasoning*, Athens.

Seaford R. (2004), *Money and the Early Greek Mind*, Cambridge.

Snell B. (1944), "Die Nachrichten über die Lehren des Thales und die Anfänge der griechischen Philosophie- und Literaturgeschichte," *Philologus*, 96, p. 170-182.

Thonemann P. (2011), *The Meander Valley*, Cambridge.

Unguru S. (1975), "On the need to Rewrite the History of Greek Mathematics," *Archiv for the History of Exact Sciences*, 15, p. 67-114.

Zhmud L. (2012), *Pythagoras and the Pythagoreans*, Cambridge.

Zhmud L. (2007), "Review: Daniel Graham, *Explaining the Cosmos*," *Gnomon*, 80/6, p. 1-5.